Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Richard Wünsch

The gravitational instability of expanding HI shells

Astronomický ústav Univerzity Karlovy

Vedoucí diplomové práce: Doc. RNDr. Jan Palouš, DrSc.
Studijní program: Fyzika, Astronomie a astrofyzika

Děkuji Doc. Janu Paloušovi za neocenitelnou pomoc při vytváření této práce, za mnoho konstruktivních připomínek a především za čas a trpělivost, kterou mi věnoval.

Děkuji také Astronomickému ústavu AV ČR za poskytnutí hardwaru a softwaru pro počítačové simulace.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 13. 4. 2000

Richard Wünsch

Contents

Introduction

The purpose of this work is to include the non-linear terms to the analysis of the fragmentation of expanding HI shells. It is an extension of previous linear analysis which was performed by Elmegreen (1994). We use the similar approach as Fuchs (1996) who described the fragmentation of uniformly rotating self-gravitating discs.

If some conditions are fulfilled, the HI shell may become gravitationally unstable and then it may break to fragments. Inclusion of higher order terms to the analysis can help us to determine with better accuracy the time when the shell is fragmented than the linear analysis.

The thesis is divided in seven sections. In the first and the second section are HI shells introduced, in the third section some models of the HI shells are described, in the fourth section are derived hydrodynamical equations for the cold and thin shell, which is expanding into the uniform ambient medium. In the fifth section the linear analysis of the shell instability is presented. The criteria for the shell instability is the same as in (Elmegreen 1994), but it was derived by another formalism. In the seventh section are nonlinear equations solved, and it is shown that the interaction of the perturbation modes of the shell surface density occurs, similar to the case of the rotating self-gravitating disc (Fuchs, 1996).

In the appendix the complete derivation of non-linear equations is given.

1 General information

1.1 What are HI shells

HI shells are structures in the neutral hydrogen in galaxies. They are regions filled by the hot gas of the low density surrounded by the dense thin shell. The scales of these shells are from 10 pc to $1-2$ kpc. The typical density in the cavity is $10^{-3} - 10^{-4}$ hydrogen atoms per cm^3, the typical surface density of the shell is $10^{20} - 10^{21}$ hydrogen atoms per cm^2. In the most common concept HI shells are the result of the shock wave in the ISM created by a very energetic explosion. For instance the shock wave after the SN explosion sweeps up the ambient medium and by this way it creates a cavity filled by the hot gas surrounded by the cold dense thin shell.

1.2 Discovery of HI shells and supershells

First observations of HI shells in the Milky Way appear more than 40 years ago (Menon, 1958). Later, many shells and supershells were discovered by Heiles (1979) in the Weaver and Williams (1973) HI survey of the Milky Way. HI shells were also observed in LMC, SMC and other nearby galaxies.

1.3 Origin of HI shells

The typical energy needed to create a shell or supershell is $10^{51} - 10^{54}$ $ergs$. It means that single SN with the energy 10^{51} $ergs$ can be responsible only for the smallest structures. Larger shells and supershells can be created by an OB association with $10 - 100$ massive stars. These stars have short live time and finally explode as supernovae, so they can add with the stellar wind and SN explosions the energy to the cavity. The most energetic shells ($10^{53} - 10^{56} ergs$) may be formed by the group of several OB associations. Alternatively, HI shells can be formed by encounter of the galactic HI plane with a high velocity cloud or a dwarf galaxy. Recently, also explosions connected to GRB are considered.

2 Why are HI shells interesting

2.1 Propagating star formation

During the expansion into the ambient medium, new mass is accreted to the shell while its expansion velocity and temperature are decreasing. In the typical density of the ambient medium 1 cm^{-3} the swept up mass is between $10^4 - 10^7\ M_\odot$. The shell may become gravitationally unstable and fragment and form new molecular clouds, in which new stars may form. By this mechanism HI shells can propagate the star formation.

2.2 Disc halo connection

In galaxies with thin gaseous discs, for instance in rapidly rotating spiral galaxies, the shape of the shell is strongly affected by the z-distribution of the gas: cylindrical, z-elongated structures called *worms* may form. The worm can break through the disc and even open to the galactic halo. These ones are called *blow-out* shells and *chimneys*.

In case of a blow-out shell, the hot gas from the shell interior enriched by metals made by stars escapes to the halo. It can be the way how to explain the observed hot metal rich halo gas (Cox & Smith, 1974; McKee & Ostriker, 1977).

3 Models of HI shells

3.1 Sedov solution

The simplest model of the expanding shell was invented by Sedov (1959). The infinitesimally thin shell approximation is considered in a solution of equation of motion of a strong shock propagating into the ambient medium. The equation of motion has form:

$$\frac{d}{dt}(mV) = S(P_{int} - P_{ext})\ , \tag{1}$$

where m is the mass of the shell, V is expansion velocity, S is surface of the shell and P_{int} and P_{ext} are the pressures inside and outside of the shell. In the solution of Sedov (1959) the external pressure is neglected. In case of spherical symmetry and abrupt energy input have the solutions for radius of the shell R and expansion velocity V form:

$$\begin{aligned} R &\sim E_0^{1/5} \cdot n_0^{-1/5} \cdot t^{2/5} \\ V &\sim E_0^{1/5} \cdot n_0^{-1/5} \cdot t^{-3/5}\ , \end{aligned} \tag{2}$$

where E_0 is the input energy, n_0 is density of the ambient medium and t is the time since the beginning of the expansion. In case of cylindrical symmetry the solutions have form:

$$\begin{aligned} R &\sim E_0^{1/4} \cdot n_0^{-1/4} \cdot t^{1/2} \\ V &\sim E_0^{1/4} \cdot n_0^{-1/4} \cdot t^{-1/2}\ , \end{aligned} \tag{3}$$

and in case of spherical symmetry and continuous energy input have the solutions form:

$$\begin{aligned} R &\sim (\dot{N}_{SN} \cdot E_{SN})^{1/5} \cdot n_0^{-1/5} \cdot t^{3/5} \\ V &\sim (\dot{N}_{SN} \cdot E_{SN})^{1/5} \cdot n_0^{-1/5} \cdot t^{-2/5} \ , \end{aligned} \tag{4}$$

where $\dot{N}_{SN}$ is the supernova rate and E_{SN} is energy of one supernova.

3.2 Chevalier's formula

The relation which enable us to estimate the energy of the observed shell was obtained by Chevalier (1974) from one dimensional hydrodynamical simulations. He used model of the spherically symmetric shell expanding into uniform medium. The external pressure and cooling were considered.

His computation is resulted in Chevalier's formula:

$$\left(\frac{E_0}{10^{50}\ erg}\right) = 5.3 \cdot 10^{-7} \left(\frac{n_0}{cm^{-3}}\right)^{1.12} \left(\frac{v}{km\ s^{-1}}\right)^{1.4} \left(\frac{R}{pc}\right)^{3.12} \ . \tag{5}$$

3.3 Other computer simulations

Two dimensional computer simulations with the thin shell approximation have been performed by Mac Low & McCray (1988) and Mac Low et al. (1989) using the hydrodynamical code ZEUS. The ambient medium stratified in the direction perpendicular to the galactic plane was considered and the possibility of the creation of blow-out shells was shown.

Other computer simulations in 2D using the thin shell approximation have been performed by Tenorio-Tagle & Palouš (1987). This approximation is also called 1+1/2 dimensional, which includes the galactic differential rotation.

The 2+1/2 dimensional models (thin shell approximation in 3D) have been developed by Palouš (1990, 1992), Silich (1996) and Ehlerová et. al (1997). Beyond the z-stratification of ambient medium and galactic differential rotation they include also cooling and evaporation of small preexisting interstellar clouds.

4 Basic equations

We consider a cold and thin shell of radius R in three-dimensional space with hot interior expanding with velocity V into a uniform medium of density ρ_0. The intrinsic surface density of the shell σ is composed of unperturbed part

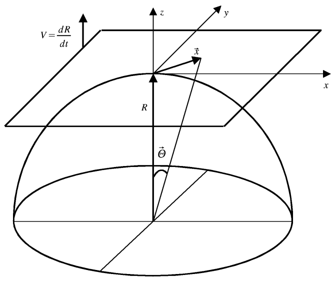

Figure 1: *The definition of the coordinates. On the shell surface are used angular coordinates* $\vec{\Theta} = \vec{x}/R$ *for the position and the angular velocity* $\vec{\Omega} = \vec{v}/R$ *for the surface flows, where* $\vec{v}$ *is a normal 2D velocity.*

σ_0 plus the perturbation σ_1 ($\sigma = \sigma_0 + \sigma_1$). Perturbation σ_1 results from the flows on the surface of the shell redistributing the accumulated mass. We assume that σ_0 corresponds to R as $\sigma_0 = \rho_0 R/3$, which means that all the encountered mass is accumulated to the shell. (It comes from $\frac{4}{3}\pi R^3 \rho_0 / 4\pi R^2$).

We can write the mass conservation law for a small area $A = 4\pi\alpha R^2$ of the shell, where α is angular size of the area.

$$\frac{\partial M}{\partial t} + \nabla \cdot M\vec{v} = 0, \tag{6}$$

where M is total mass in the area, $\vec{v}$ denotes a two-dimensional velocity of surface flows: $\vec{x}$ and $\vec{v} = \dot{\vec{x}}$ are two-dimensional vectors in the tangential plane of the shell at the central point of the area A. We consider angular coordinates $\vec{\Theta} = \vec{x}/R$ and angular velocity $\vec{\Omega} = \vec{v}/R$ to describe the surface of the shell (see Fig. 1). With $M = \sigma A$ we obtain continuity equation in a form

$$\frac{\partial \sigma}{\partial t} + 2\sigma\frac{V}{R} + \sigma R \nabla \cdot \vec{\Omega} + R\vec{\Omega} \cdot \nabla\sigma = 0. \tag{7}$$

The equation of motion for the element of mass M and area A has form

$$\frac{1}{A}\frac{d(M\vec{v})}{dt} = -c^2 \nabla\sigma - \sigma\nabla\Phi, \tag{8}$$

where c is constant isothermal sound speed, Φ is gravitational potential of the shell. Assuming $M = 4\pi\alpha R^3 \rho_0/3$ and $A = 4\pi\alpha R^2$ we can write it as

$$R\frac{\partial \vec{\Omega}}{\partial t} + R^2 \vec{\Omega} \cdot \nabla\vec{\Omega} + V\vec{\Omega} + 3V\vec{\Omega}\frac{\sigma_0}{\sigma} = -\frac{c^2}{\sigma}\nabla\sigma - \nabla\Phi \tag{9}$$

The gravitational potential Φ is related to the surface density by the Poisson equation

$$\Delta\Phi = 4\pi G\sigma\delta(z), \tag{10}$$

where G is the constant of gravitation and $\delta(z)$ is a delta function of coordinate z perpendicular to the surface of the shell.

5 Perturbation analysis

We assume a small perturbation of the shell surface density $\sigma_1 \ll \sigma_0$ which evolves due to surface flows given with velocity $\vec{v}$. The related perturbation of gravitational potential Φ_1 can be derived from the Poisson equation. We rewrite the equations (7), (9) and (10) in form

$$\frac{\partial\sigma_1}{\partial t} + 2\sigma_1\frac{V}{R} + \sigma R\nabla\cdot\vec{\Omega} + R\vec{\Omega}\cdot\nabla\sigma_1 = 0, \tag{11}$$

$$R\frac{\partial\vec{\Omega}}{\partial t} + R^2\vec{\Omega}\cdot\nabla\vec{\Omega} = -\frac{c^2}{\sigma_0}(1-\frac{\sigma_1}{\sigma_0})\nabla\sigma_1 - \nabla\Phi_1 - 4V\vec{\Omega} + 3V\vec{\Omega}\frac{\sigma_1}{\sigma_0}, \tag{12}$$

$$\Delta\Phi_1 = 4\pi G\sigma_1\delta(z), \tag{13}$$

where the $1/\sigma$ in equation (9) was evaluated up to quadratic terms in σ_1.

These equations may be Fourier transformed with respect to the spatial coordinates similar to Fuchs (1996)

$$\begin{aligned}\sigma_1 &= \sigma_{10} + \sum_{\vec{\eta}}\sigma_{\vec{\eta}}e^{i\vec{\eta}\cdot\vec{\Theta}}\\ \vec{\Omega} &= \vec{\Omega}_0 + \sum_{\vec{\eta}}\vec{\Omega}_{\vec{\eta}}e^{i\vec{\eta}\cdot\vec{\Theta}}\end{aligned} \tag{14}$$

where $\vec{\eta}$ denotes a dimensionless wavevector $\vec{\eta} = \vec{k}R$. We assume no surface macroscopic flow through the all considered area which means $\vec{\Omega}_0 = 0$. Further we assume $\sigma_{10} = 0$ (mass accumulation due to expansion to the ambient medium is included in σ_0). The Fourier transform of the equation (11) is

$$\dot{\sigma}_{\vec{\eta}} + \sigma_0(i\vec{\eta}\cdot\vec{\Omega}_{\vec{\eta}}) + 2\frac{V}{R}\sigma_{\vec{\eta}} + \sum_{\vec{\eta}'}(\vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\cdot i\vec{\eta})\sigma_{\vec{\eta}'} = 0 \tag{15}$$

where $\vec{\eta}' + (\vec{\eta} - \vec{\eta}') = \vec{\eta}$ was used. The Fourier transform of Euler's equation (12) is

$$R\dot{\vec{\Omega}}_{\vec{\eta}} + R\sum_{\vec{\eta}'}(\vec{\Omega}_{\vec{\eta}-\vec{\eta}'} \cdot i\vec{\eta}')\vec{\Omega}_{\vec{\eta}'} = -\frac{c^2}{\sigma_0} i\vec{k}\sigma_{\vec{\eta}} + \frac{c^2}{\sigma_0^2}\sum_{\vec{\eta}'} ik'\sigma_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'}$$

$$+2\pi G\frac{i\vec{\eta}}{|\vec{\eta}|}\sigma_{\vec{\eta}} - 4V\vec{\Omega}_{\vec{\eta}} + \frac{3V}{\sigma_0}\sum_{\vec{\eta}'}\vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'}, \tag{16}$$

where the following solution of Poisson equation was used

$$\nabla\Phi_1 = -2\pi G\sum_{\vec{\eta}}\sigma_{\vec{\eta}}\frac{i\vec{\eta}}{|\vec{\eta}|}e^{i\vec{\eta}\cdot\vec{\Theta}}. \tag{17}$$

6 Linear analysis

6.1 The instability criteria

Linearized equations (15) and (16) have form

$$\dot{\sigma}_{\vec{\eta}} + \sigma_0(i\vec{\eta}\cdot\vec{\Omega}_{\vec{\eta}}) + 2\frac{V}{R}\sigma_{\vec{\eta}} = 0 \tag{18}$$

$$R\dot{\vec{\Omega}}_{\vec{\eta}} = -\frac{c^2}{\sigma_0}i\vec{k}\sigma_{\vec{\eta}} + 2\pi G\frac{i\vec{\eta}}{|\vec{\eta}|}\sigma_{\vec{\eta}} - 4V\vec{\Omega}_{\vec{\eta}}. \tag{19}$$

Angular velocity $\vec{\Omega}_{\vec{\eta}}$ can be split in two components parallel and orthogonal to the wavevector $\vec{\eta}$

$$\vec{\Omega}_{\vec{\eta}} = \Omega_{\vec{\eta}_{\|}}\frac{\vec{\eta}_{\|}}{\eta_{\|}} + \Omega_{\vec{\eta}_{\perp}}\frac{\vec{\eta}_{\perp}}{\eta_{\perp}}. \tag{20}$$

We get the set of equations

$$\dot{\sigma}_{\vec{\eta}} = -2\frac{V}{R}\sigma_{\vec{\eta}} - i\eta\sigma_0\Omega_{\vec{\eta}_{\|}} \tag{21}$$

$$\dot{\Omega}_{\vec{\eta}_{\|}} = (i\frac{2\pi G}{R} - i\frac{c^2\eta}{\sigma_0 R^2})\sigma_{\vec{\eta}} - 4\frac{V}{R}\Omega_{\vec{\eta}_{\|}} \tag{22}$$

$$\dot{\Omega}_{\vec{\eta}_{\perp}} = -4\frac{V}{R}\Omega_{\vec{\eta}_{\perp}} \tag{23}$$

To solve the set of equations (21) - (23) we assume a time dependence

$$\sigma_{\vec{\eta}},\ \Omega_{\vec{\eta}_{\|}},\ \Omega_{\vec{\eta}_{\perp}} \sim e^{i\omega t}\ . \tag{24}$$

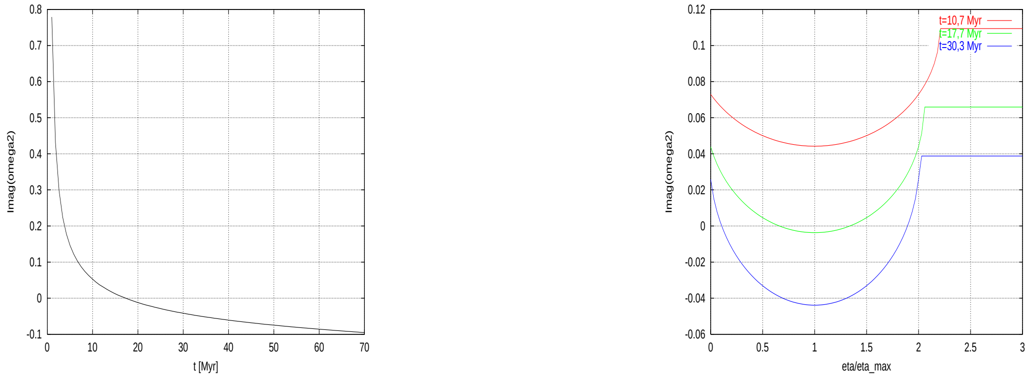

Figure 2: *Left: The time dependence of the imaginary part of the $\omega^{(2)}$, which can cause the instability. Right: The imaginary part of the same $\omega^{(2)}$ depending on the wavevector η for three times. If the imaginary part of $\omega^{(2)}$ is negative, the shell is unstable. The Sedov solution was used with following parameters: total energy $E_{tot} = 10^{53} erg$, density of ambient medium $n_0 = 1\ cm^{-3}$, average molecular weight $\mu = 1.3$, sound speed in the shell $c = 1\ km \cdot s^{-1}$.*

We get the eigenvalues

$$\omega_{\vec{\eta}}^{(1,2)} = i\frac{3V}{R} \pm \sqrt{-\frac{V^2}{R^2} + \frac{\eta^2 c^2}{R^2} - \frac{2\pi G \sigma_0 \eta}{R}} \tag{25}$$

$$\omega_{\vec{\eta}}^{(3)} = i4\frac{V}{R}\ . \tag{26}$$

The $\omega^{(1,2)}$ in equation (25) are almost the same as in equation (6) in Elmegreen (1994). The related eigenvectors are

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{||}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1,2)} = \begin{pmatrix} -i\eta\sigma_0 \\ i\omega^{(1,2)} + 2\frac{V}{R} \\ 0 \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{||}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\ . \tag{28}$$

The $\omega_{\vec{\eta}}$ are time dependent, so the assumption (24) is fulfilled only in the short time interval, in which $\omega_{\vec{\eta}}(t)$ changes not too much. But it can be used as a criteria of the shell's instability by the following way. The $\omega_{\vec{\eta}}^{(3)}$ has always meaning of the decrease of perturbations. If $\omega_{\vec{\eta}}^{(1,2)}$ have got a real part, solution is stable with decreasing oscillations. If not, $\omega_{\vec{\eta}}^{(1)}$ indicates decrease, $\omega_{\vec{\eta}}^{(2)}$ can be imaginary negative and it have meaning of the growth of perturbations. The time evolution of the imaginary part of the $\omega_{\vec{\eta}}^{(2)}$ is shown by figure 2.

The eigenvalues $\omega_{\vec{\eta}}$ depend also on the magnitude of the wavevector $\vec{\eta}$ as suggested by figure 2. The η with the maximum perturbation growth rate can be found. This was done by Elmegreen (1994).

The maximum perturbation growth rate is:

$$\omega_{\vec{\eta},max}^{(1,2)} = i\frac{3V}{R} \pm \sqrt{-\frac{V^2}{R^2} - \frac{\pi^2 G^2 \sigma_0^2}{c^2}} \tag{29}$$

and occurs at the wavenumber:

$$\eta_{max} = \frac{\pi G \sigma_0 R}{c^2} \ . \tag{30}$$

6.2 The time of the fragmentation

If the shell is unstable, i. e. the imaginary part of the $\omega^{(2)}$ is negative, for the certain time, the shell may break to fragments. In Ehlerová et al. (1997) the *fragmentation integral* is defined as:

$$I_{frag}(t) \equiv \int_{t_b}^{t} \omega_{\eta,max}^{(2)}(t')dt' \ , \tag{31}$$

where t_b is the time when the instability begins. The fragmentation time $t_{f,l}$ (the time when the shell is decomposed to fragments) is defined as the time when the fragmentation integral is equal to one:

$$I_{frag}(t_f, l) = \int_{t_b}^{t_f} \omega_{\eta,max}^{(2)}(t')dt' = 1 \ . \tag{32}$$

7 Non-linear analysis

7.1 The non-linear equations

We rewrite the non-linear equations (15) and (16) in the form

$$\begin{pmatrix} \dot{\sigma}_{\vec{\eta}} \\ \dot{\Omega}_{\vec{\eta}_{\parallel}} \\ \dot{\Omega}_{\vec{\eta}_{\perp}} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix} + \mathcal{N} \ , \tag{33}$$

where $\mathcal{L}$ is the linear part and $\mathcal{N}$ represents the non-linear terms. We search for a solution of equations (33) as a combination of the eigenvectors obtained from the previous linear analysis

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix} = \psi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} + \xi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} + \phi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} , \tag{34}$$

where $\psi_{\vec{\eta}(t)}$, $\xi_{\vec{\eta}(t)}$ and $\phi_{\vec{\eta}(t)}$ are time dependent amplitudes of the eigenvectors. We find orthonormal vectors $(\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(1,2,3)}$ in order that

$$(\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(i)} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(i)} = 1, \qquad (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(i)} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(j \neq i)} = 0 \, ,$$

where $i = 1, 2, 3$. The orthonormal vectors are

$$\begin{array}{l} (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(1)} = \left(\frac{i\omega^{(2)} + 2\frac{V}{R}}{\eta\sigma_0(\omega^{(1)} - \omega(2))}, \frac{i}{(\omega^{(1)} - \omega(2))}, 0 \right) \\ (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(2)} = \left(\frac{i\omega^{(1)} + 2\frac{V}{R}}{\eta\sigma_0(\omega^{(2)} - \omega(1))}, \frac{i}{(\omega^{(2)} - \omega(1))}, 0 \right) \\ (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(3)} = (0, 0, 1) \end{array} . \tag{35}$$

We insert ansatz (34) into equation (33), multiply it by the orthonormal vectors (35) and obtain a set of equations for amplitudes $\psi_{\vec{\eta}(t)}$, $\xi_{\vec{\eta}(t)}$ and $\phi_{\vec{\eta}(t)}$

$$\begin{aligned} \dot{\psi}_{\vec{\eta}} = i\omega^{(1)}\psi_{\vec{\eta}} + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} \cdot (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(1)} \psi_{\vec{\eta}} + \\ + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} \cdot (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(1)} \xi_{\vec{\eta}} + (\mathcal{N}, (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(1)}) \end{aligned} \tag{36}$$

$$\begin{aligned} \dot{\xi}_{\vec{\eta}} = i\omega^{(2)}\xi_{\vec{\eta}} + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} \cdot (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(2)} \xi_{\vec{\eta}} + \\ + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} \cdot (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(2)} \psi_{\vec{\eta}} + (\mathcal{N}, (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(2)}) \end{aligned} \tag{37}$$

$$\dot{\phi}_{\vec{\eta}} = i\omega^{(3)}\phi_{\vec{\eta}} + \frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\parallel}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} \cdot (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(3)}\phi_{\vec{\eta}} + (\mathcal{N}, (\sigma_{\vec{\eta}}, \Omega_{\vec{\eta}_{\parallel}}, \Omega_{\vec{\eta}_{\perp}})^{(3)}) \quad (38)$$

The most interesting is equation (37), because it has $\omega^{(2)}$ in the first linear term, and only the $\omega^{(2)}$ can be imaginary negative, which has meaning of instability. Equations (36) and (38) have in the first linear term "stable" $\omega^{(1)}$ and $\omega^{(3)}$. The second linear term (with time derivatives) and the nonlinear terms are too small against the first linear term. The solutions of equations (36) and (38) have decreasing or oscilating character.

Using (34) we get an explicit form of the equation (37)

$$\begin{aligned}
\dot{\xi}_{\vec{\eta}} \;=\;& i\omega^{(2)}\xi_{\vec{\eta}} - \frac{i(\dot{\eta}\sigma_0 + \eta\dot{\sigma}_0)(i\omega^{(1)} + 2\frac{V}{R}) + i\eta\sigma_0(i\dot{\omega}^{(2)} + 2\frac{\dot{V}R - V^2}{R^2})}{\eta\sigma_0(\omega^{(1)} - \omega^{(2)})}\xi_{\vec{\eta}} \\
& - \frac{i(\dot{\eta}\sigma_0 + \eta\dot{\sigma}_0)(i\omega^{(1)} + 2\frac{V}{R}) + i\eta\sigma_0(i\dot{\omega}^{(2)} + 2\frac{\dot{V}R - V^2}{R^2})}{\eta\sigma_0(\omega^{(2)} - \omega^{(2)})}\psi_{\vec{\eta}} \\
& + \frac{\omega^{(1)} - i2\frac{V}{R}}{\eta\sigma_0\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right) \right.\right. \\
& \left.\left. + i\left(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'}\right)\right] \frac{(\vec{\eta} - \vec{\eta}', \vec{\eta})}{|\vec{\eta} - \vec{\eta}'|} + \phi_{\vec{\eta}-\vec{\eta}'} \frac{(\vec{\eta} - \vec{\eta}'_{\perp}, \vec{\eta})}{|\vec{\eta} - \vec{\eta}'_{\perp}|} \right\} \\
& \left[-i\eta'\sigma_0\left(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'}\right)\right] + \frac{1}{\omega^{(1)} - \omega^{(2)}} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right) \right.\right. \\
& \left.\left. + i(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'})\right] \frac{(\vec{\eta} - \vec{\eta}', \vec{\eta}')}{|\vec{\eta} - \vec{\eta}'|} + \phi_{\vec{\eta}-\vec{\eta}'} \frac{(\vec{\eta} - \vec{\eta}'_{\perp}, \vec{\eta}')}{|\vec{\eta} - \vec{\eta}'_{\perp}|} \right\} \\
& \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'}\right) + i\left(\omega^{(1)}\psi_{\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}'}\right)\right] \frac{(\vec{\eta}', \vec{\eta})}{|\vec{\eta}'|\,|\vec{\eta}|} + \phi_{\vec{\eta}'} \frac{(\vec{\eta}'_{\perp}, \vec{\eta}')}{|\vec{\eta}'_{\perp}|\,|\vec{\eta}|} \right\} \\
& - \frac{c^2}{\sigma_0^2 R^2\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left[-i\left(\eta - \eta'\right)\sigma_0\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right)\right] \\
& \left[-i\eta'\sigma_0(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'})\right] \frac{(\vec{\eta}', \vec{\eta})}{|\vec{\eta}|} + \frac{i3V}{R\sigma_0\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R} \right.\right. \\
& \left. (\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}) + i(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'})\right] \frac{(\vec{\eta} - \vec{\eta}', \vec{\eta})}{|\vec{\eta} - \vec{\eta}'|\,|\vec{\eta}|} \\
& \left. + \phi_{\vec{\eta}-\vec{\eta}'} \frac{(\vec{\eta} - \vec{\eta}'_{\perp}, \vec{\eta})}{|\vec{\eta} - \vec{\eta}'_{\perp}|\,|\vec{\eta}|} \right\} \left[-i\eta'\sigma_0(\mu_{\vec{\eta}'} + \nu_{\vec{\eta}'})\right] \qquad (39)
\end{aligned}$$

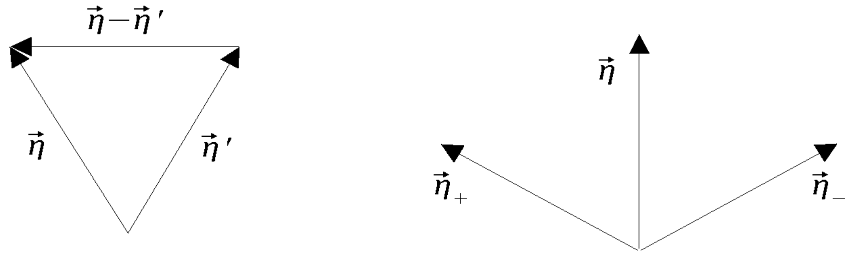

Figure 3: *Left: If* $\eta = \eta' = |\vec{\eta} - \vec{\eta}'| = \eta_{max}$*, the wavevectors build an equilateral triangle. Right: The mode* $\vec{\eta}$ *interacts with two modes inclined at an angle* 60^o *to the wavevector* $\vec{\eta}$*.*

7.2 The interaction of modes

In analogy to Fuchs (1996) we group together components with fully imaginar $\omega_{\vec{\eta}}$ into wavepackets and take the wavenumber η_{max}. Other components are grouped to the wavepackets of the same width and approximated by the average wavenumbers. The $\xi_{\vec{\eta}_{max}}$ modes grow, the $\psi_{\vec{\eta}_{max}}$ and the $\phi_{\vec{\eta}_{max}}$ modes descend. Other modes oscillate with decrease. The descending modes couple to the $\xi_{\vec{\eta}_{max}}$ through non-linear terms, they lead to the third order terms, and they may be neglected against terms of second order in $\xi_{\vec{\eta}_{max}}$.

If we demand $\eta = \eta' = |\vec{\eta} - \vec{\eta}'| = \eta_{max}$ we obtain from geometry (see fig. 3)

$$\begin{array}{ll} (\vec{\eta}, \vec{\eta}'_{\pm}) = \frac{1}{2}\eta^2_{max} & (\vec{\eta}, \vec{\eta} - \vec{\eta}'_{\pm}) = \frac{1}{2}\eta^2_{max} \\ (\vec{\eta}'_{+}, \vec{\eta} - \vec{\eta}'_{+}) = -\frac{1}{2}\eta^2_{max} & (\vec{\eta}'_{-}, \vec{\eta} - \vec{\eta}'_{-}) = -\frac{1}{2}\eta^2_{max} \end{array}, \tag{40}$$

where $\vec{\eta}'_+$ and $\vec{\eta}'_-$ are two wavevectors inclined at angles 60^o to the wavevector $\vec{\eta}$. It means that $\xi_{\vec{\eta}}$ wavepackets non-linearly interact with others with wavevectors $\vec{\eta}'_{\pm}$. Using equation (25) and geometrical considerations (40), the set of equations (39) can be simplified to the form

$$\begin{aligned} \dot{\xi} &= i\omega^{(2)}\xi + \alpha\xi - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi_+\xi_- \\ \dot{\xi}_+ &= i\omega^{(2)}\xi_+ + \alpha\xi_+ - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi\xi_-^* \\ \dot{\xi}_- &= i\omega^{(2)}\xi_- + \alpha\xi_- - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi\xi_+^* \end{aligned} \tag{41}$$

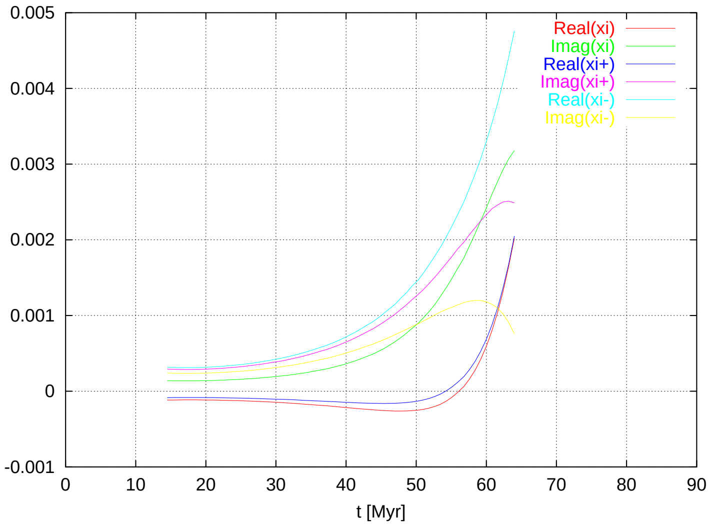

Figure 4: *The solution of the set of equations (41). The initial values were chosen randomly with gaussian dispersion which corresponds to perturbation of the surface density* $\sigma_1 = 1,4 \cdot 10^{19}\ cm^{-2}$. *The Sedov solution was used with following parameters: total energy* $E_{tot} = 10^{53}$ *erg, density of ambient medium* $n_0 = 1\ cm^{-3}$, *average molecular weight* $\mu = 1.3$, *sound speed in the shell* $c = 1\ km \cdot s^{-1}$.

where

$$\Sigma = \sqrt{-\frac{V^2c^2 + \pi^2G^2\sigma_0^2R^2}{R^2c^2}} \tag{42}$$

and

$$\alpha = \frac{-i2\dot{\sigma}_0R^2Vc^2\Sigma - 2\dot{\sigma}_0R^3c^2\Sigma^2 - \sigma_0R^2Vc^2\Sigma^2 - i\sigma_0R^2\dot{V}c^2\Sigma}{2\sigma_0R^3c^2\Sigma} + \\ + \frac{\sigma_0RV\dot{V}c^2 + \sigma_0V^3c^2 + \sigma_0^2\dot{\sigma}_0\pi^2G^2R^3}{2\sigma_0R^3c^2\Sigma} \tag{43}$$

The set of equations (41) decribe the time evolution of one triplet of the most interacting modes $(\vec{\eta}, \vec{\eta}_+, \vec{\eta}_-)$.

7.3 The numerical solution

The set of equations (41) can be solved numerically. We start at the time t_b which is the time when the instability begins (imaginary part of $\omega^{(2)}_{\vec{\eta},max}$ starts to be negative).

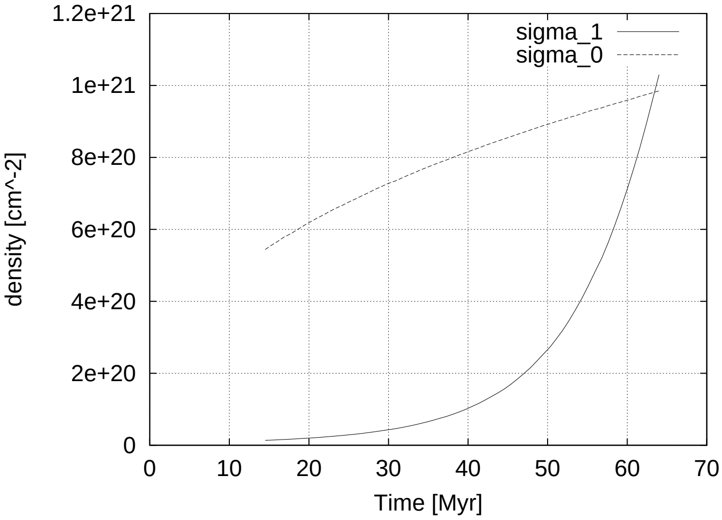

Figure 5: *The evolution of the maximum perturbation of the surface density in the case of solution presented in figure (4).*

Initial conditions are chosen randomly with the gaussian dispersion. It has meaning of initial perturbations of the surface density and the velocity. The magnitude of these perturbations in physical values can be computed from the eigenvectors (27).

The solution is determined by paramaters of two types. The first ones are constant values (as speed of sound c in the shell) which can be chosen without any problems. The second ones are functions of the time (the radius of the shell $R(t)$, the expansion velocity $V(t)$ and the surface density $\sigma_0(t)$). We can obtain them from the Sedov solution (2) or (4) or from the computer simulation of the shell.

We use the Runge-Kutta of the fourth order with variable step for the numerical integration. The typical solution is presented in figure 4. Figure (5) shows the appropriate evolution of the maximum perturbation of the surface density.

With this solution, eigenvectors (27) and equation (16) written for modes $(\vec{\eta}, \vec{\eta}_+, \vec{\eta}_-)$ only, we can get the surface density and the velocity field in a part of the shell. It was done and the result is shown by figures (7) and (8). These figures show the one triplet of the interacting modes.

In reality, there can be more triplets of the interacting modes. Figures (9) and (10) show the similar situation, but two triplets of the interacting modes inclined at angle 30^o were combined.

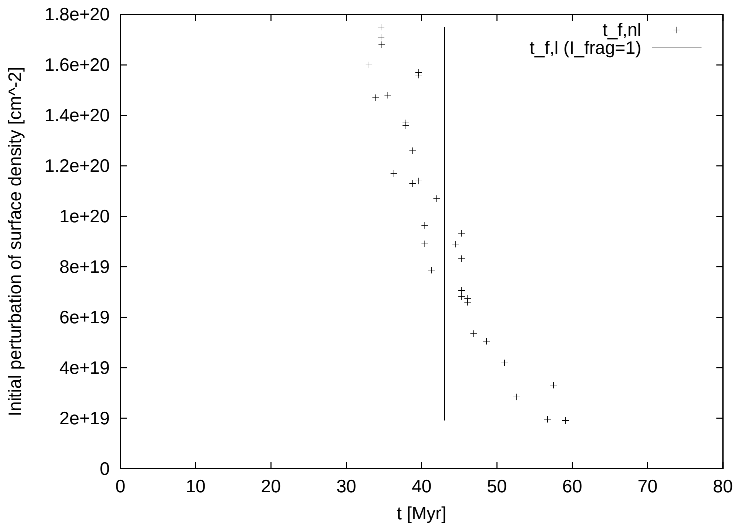

Figure 6: *Dependence of the fragmentation time on initial perturbation of the surface density.*

7.4 The time of fragmentation in the nonlinear theory

The evolution of the maximum perturbation of the surface density can be used to determine the fragmentation time of the shell. Because at advanced stages of the fragmentation the value of the maximum perturbation rises steeply with the time (see figure 5), we can define it as the time, when maximum perturbation of the surface density is equal to unperturbed value ($\sigma_1(t_{f,nl}) = \sigma_0(t_{f,nl})$).

The fragmentation time $t_{f,nl}$ defined by that way depends on initial conditions of the set of equations (41). They correspond to the initial perturbation of the surface density. We can set them to the value typical for the inhomogenities in the clumpy interstellar medium ($10^{19} - 10^{20}\ cm^{-2}$).

In figure (6) we show the dependence of the fragmentation time obtained from the solution of the non-linear equations on the value of the initial perturbation. This time may be compared to the fragmentation time obtained from the linear analysis defined as a time when $I_{frag} = 1$.

Conclusions

An interaction of modes occurs on the shell surface. It is the interaction of three modes which are inclined at the angle 60^o and is of the type discussed in Fuchs (1996).

Fragments form a few of tenth Myr after time when an instability begins.

Time when fragments form depends on initial conditions. If the initial perturbation of surface density is of the order $\sim 10^{19} - 10^{20}\ cm^{-2}$, which is a typical value of inhomogenities in the ISM, the fragmentation time is approximately of the same value as the time when the fragmentation integral $I_{frag} = \int_{t_b}^{t} \omega(t')dt'$ (which was defined in Ehlerová et al., 1997, t_b is time when the instability begins) is equal to 1.

Acknowledgment: We would like to thank Burkhard Fuchs. This work was inspired by his paper on the fragmentation of uniformly rotating discs (Fuchs, 1996). We are also greatful for an enlighting discussion in Heidelberg in March 2000.

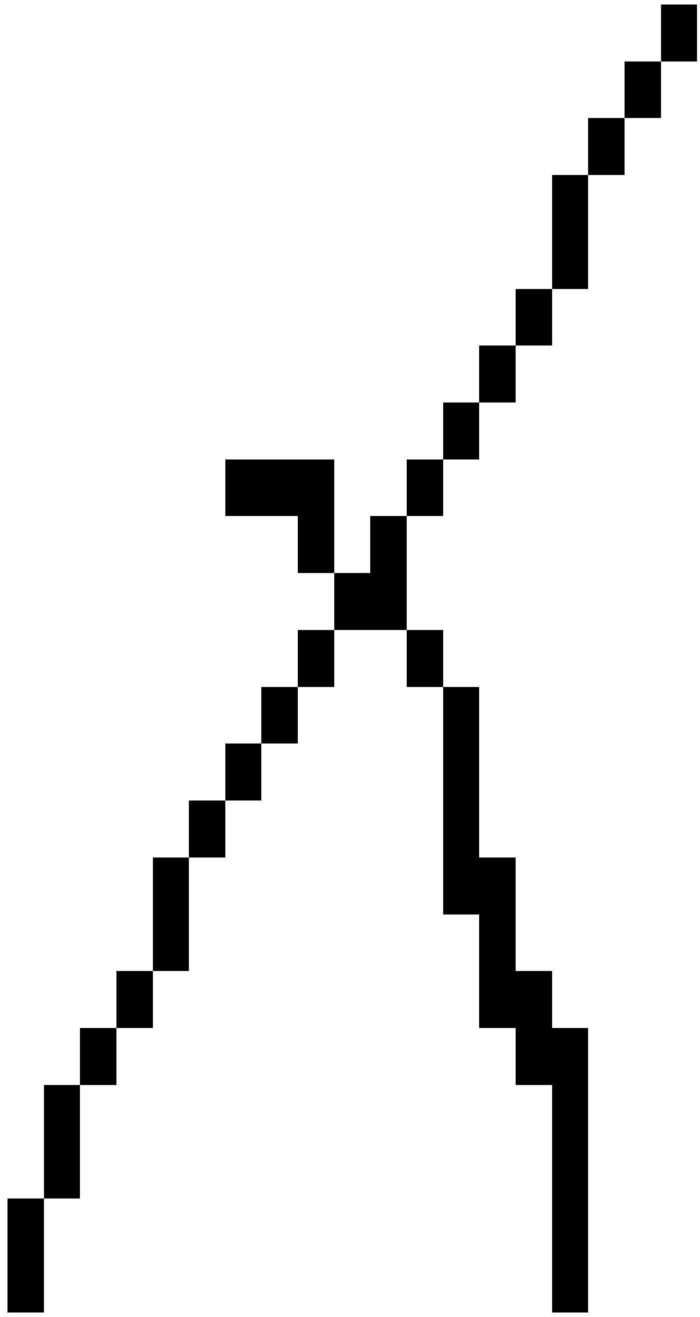

Figure 7: *The density evolution on the shell surface. One triplet of interacting modes is considered.*

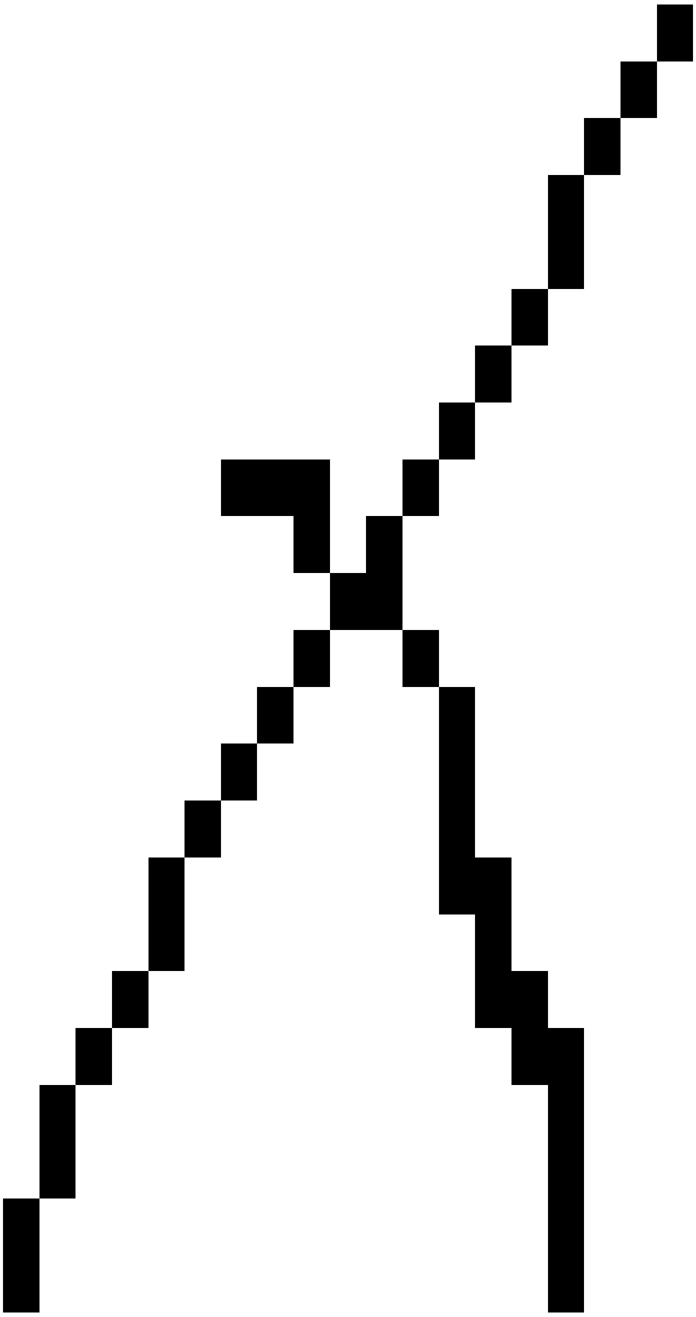

Figure 8: *The evolution of the velocity field on the shell surface. One triplet of interacting modes is considered.*

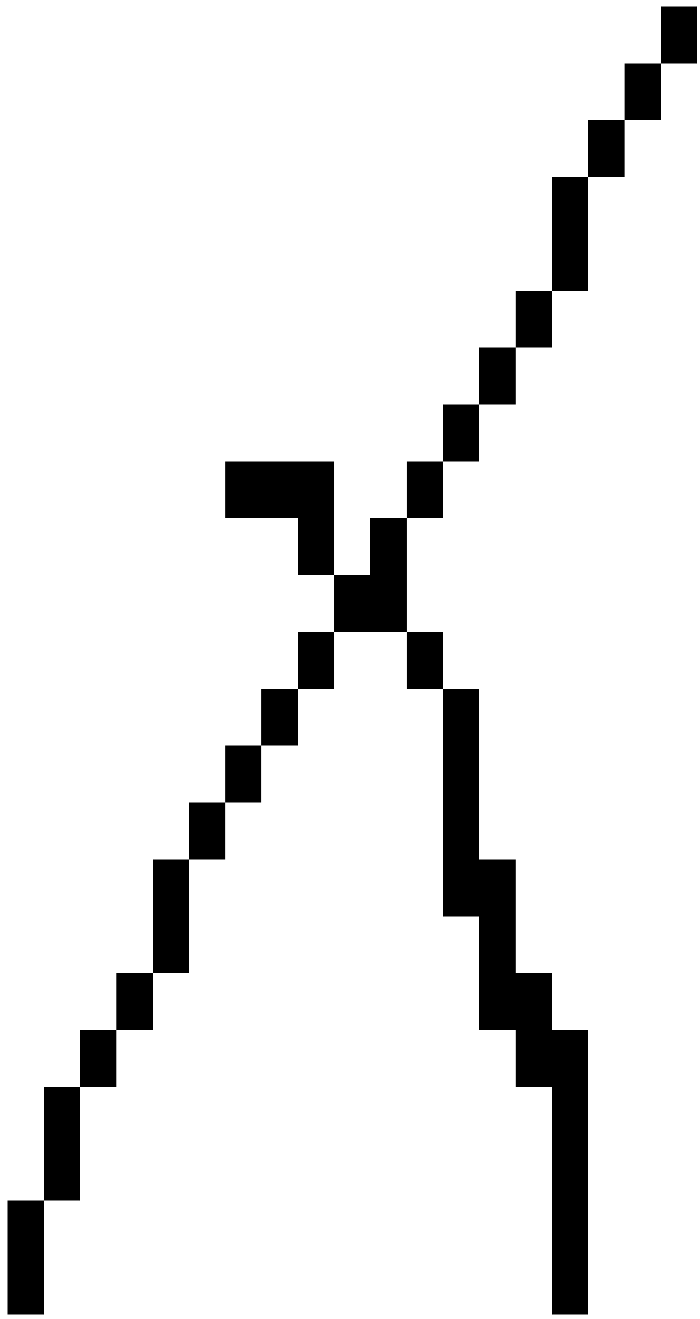

Figure 9: *The density evolution on the shell surface. Two triplets of interacting modes are considered.*

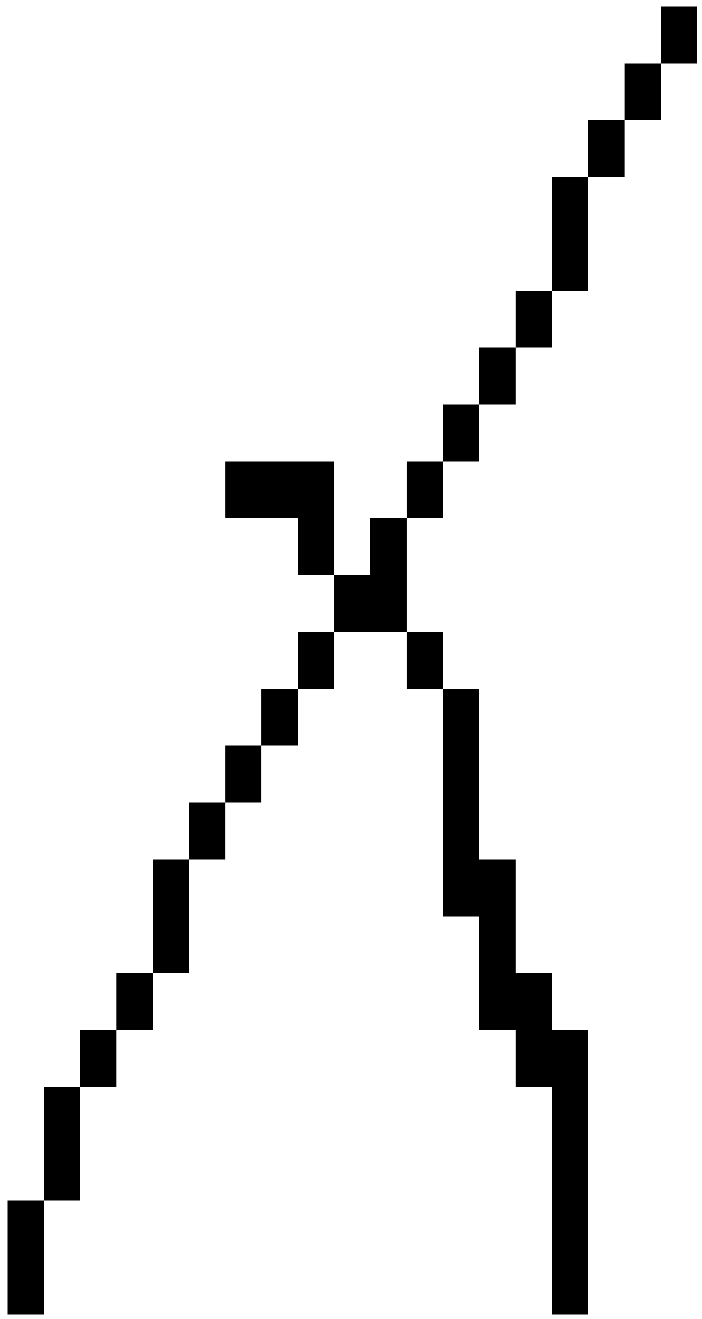

Figure 10: *The evolution of the velocity field on the shell surface. Two triplets of interacting modes are considered.*

References

[1] Chevalier R. 1974, ApJ, 188, 501

[2] Cox D. P., Smith B. W. 1974, ApJ, 89, 89, L405

[3] Ehlerová S, Palouš J., Theis Ch., Hensler G. 1997, A&A, 328, 121

[4] Elmegreen B. G. 1994, ApJ, 427, 384

[5] Fuchs B. 1996, MNRAS, 278, 985

[6] Heiles C. 1979, ApJ, 229, 533

[7] Menon T. K 1958, ApJ, 127, 28

[8] McKee C. F., Ostriker J. P. 1977, ApJ, 218, 148

[9] McLow M. M., McCray R. 1988, ApJ, 324, 776

[10] McLow M. M., McCray R., Norman M. L. 1989, ApJ, 337, 141

[11] Palouš J. 1990, in 'The Interstellar Disc-Halo Connection in Galaxies', Blomen H. (ed.), Sterrewacht Leiden, p. 101

[12] Palouš J. 1992, in 'Evolution of Interstellar Medium and Dynamics of Galaxies', Palouš J., Burton W. B., Linblad P. O. (eds.), Cambridge Univ. Press, p. 117

[13] Sedov L. I. 1959, Similarity and Dimensional Methods in Mechanics, Academy Press, New York

[14] Tenorio-Tagle G., Palouš J. 1987, A&A, 186, 287

[15] Weaver H. F., Williams D. R. 1973, A&A Suppl. 8, 1

A Derivation

A.1 Basic equations

R, V, ρ_0, $\sigma = \sigma_0 + \sigma_1$
$\sigma_0 = \rho_0 R/3$ ($= \frac{4}{3}\pi R^3 \rho_0 / 4\pi R^2$).

Continuity equation

$$\frac{\partial M}{\partial t} + \nabla \cdot M\vec{v} = 0, \tag{6}$$

$\vec{\Theta} = \vec{x}/R$, $\vec{\Omega} = \vec{v}/R$
$M = \sigma A$, $A = 4\pi\alpha R^2$

$$\frac{\partial \sigma}{\partial t} A + \sigma \frac{\partial A}{\partial t} + A\sigma \nabla \cdot R\vec{\Omega} + R\vec{\Omega} \cdot \nabla \sigma A = 0 \qquad /A$$

$$\frac{\partial \sigma}{\partial t} + 2\sigma \frac{V}{R} + \sigma R \nabla \cdot \vec{\Omega} + R\vec{\Omega} \cdot \nabla \sigma = 0. \tag{7}$$

Equation of motion

$$\frac{1}{A}\frac{d(M\vec{v})}{dt} = -c^2 \nabla \sigma - \sigma \nabla \Phi, \tag{8}$$

$M = 4\pi\alpha R^3 \rho_0 / 3$, $A = 4\pi\alpha R^2$

$$\frac{dM}{dt} = 3 \cdot 4\pi\alpha R^2 \rho_0 V$$

$$\frac{d\vec{v}}{dt} = V\vec{\Omega} + R\frac{\partial \vec{\Omega}}{\partial t}$$

$$R\frac{\partial \vec{\Omega}}{\partial t} + R^2 \vec{\Omega} \cdot \nabla \vec{\Omega} + V\vec{\Omega} + 3V\vec{\Omega}\frac{\sigma_0}{\sigma} = -\frac{c^2}{\sigma}\nabla \sigma - \nabla \Phi \tag{9}$$

Poisson equation

$$\Delta \Phi = 4\pi G \sigma \delta(z), \tag{10}$$

A.2 Perturbation analysis

Continuity equation

$$\frac{\partial \sigma}{\partial t} + 2\sigma\frac{V}{R} + \sigma R \nabla \cdot \vec{\Omega} + R\vec{\Omega} \cdot \nabla\sigma = 0.$$

$$-\Big(\frac{\partial \sigma_0}{\partial t} + 2\sigma_0\frac{V}{R} = 0\Big)$$

$$\frac{\partial \sigma_1}{\partial t} + 2\sigma_1\frac{V}{R} + \sigma R \nabla \cdot \vec{\Omega} + R\vec{\Omega} \cdot \nabla\sigma_1 = 0, \tag{11}$$

Equation of motion

$$R\frac{\partial \vec{\Omega}}{\partial t} + R^2\vec{\Omega}\cdot\nabla\vec{\Omega} + V\vec{\Omega} + 3V\vec{\Omega}\frac{\sigma_0}{\sigma} = -\frac{c^2}{\sigma}\nabla\sigma - \nabla\Phi$$

$\nabla\sigma = \nabla\sigma_1$

$$R\frac{\partial \vec{\Omega}}{\partial t} + R^2\vec{\Omega}\cdot\nabla\vec{\Omega} + V\vec{\Omega} + 3V\vec{\Omega}(1-\frac{\sigma_1}{\sigma_0}) = -\frac{c^2}{\sigma_0^2}(1-\frac{\sigma_1}{\sigma_0})\nabla\sigma_1 - \nabla\Phi$$

$$-\Big(0 = \nabla\Phi_0\Big)$$

$$R\frac{\partial \vec{\Omega}}{\partial t} + R^2\vec{\Omega}\cdot\nabla\vec{\Omega} = -\frac{c^2}{\sigma_0}(1-\frac{\sigma_1}{\sigma_0})\nabla\sigma_1 - \nabla\Phi_1 - 4V\vec{\Omega} + 3V\vec{\Omega}\frac{\sigma_1}{\sigma_0}, \tag{12}$$

Poisson equation

$$\Delta\Phi = 4\pi G\sigma\delta(z),$$

$$-\Big(\Delta\Phi_0 = 4\pi G\sigma_0\delta(z)\Big)$$

$$\Delta\Phi_1 = 4\pi G\sigma_1\delta(z), \tag{13}$$

A.3 Fourier transformation

$$\sigma_1 = \sigma_{10} + \sum_{\vec{\eta}} \sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}$$
$$\vec{\Omega} = \vec{\Omega}_0 + \sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} \tag{14}$$

$\vec{\eta} = \vec{k}R.$
We assume $\vec{\Omega}_0 = 0$, $\sigma_{10} = 0$
$\vec{\eta}' + (\vec{\eta} - \vec{\eta}') = \vec{\eta}$

Continuity equation

$$\begin{aligned}\dot{\sigma}_{10} + \sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + 2\frac{V}{R}(\sigma_{10} + \sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) +\\ +\sigma_0 R\nabla\cdot(\vec{\Omega}_{10} + \sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) +\\ +R(\sigma_{10} + \sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})\nabla\cdot(\vec{\Omega}_{10} + \sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) +\\ +R(\vec{\Omega}_{10} + \sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})\cdot\nabla(\sigma_{10} + \sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})\end{aligned}$$

$$\eta = 0 \ : \ \dot{\sigma}_{10} + 2\frac{V}{R}\sigma_{10} + R\sum_{\vec{\eta}}(\vec{\Omega}_{\vec{\eta}}\cdot(-i\vec{k}))\sigma_{-\vec{\eta}} + R\sum_{\vec{\eta}}\sigma_{\vec{\eta}}((-i\vec{k})\cdot\vec{\Omega}_{-\vec{\eta}}) = 0$$

$$\begin{aligned}\eta \neq 0 \ : \quad & \textstyle\sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + 2\frac{V}{R}\sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \sigma_0 R\nabla\cdot(\sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) +\\ & \textstyle +R\sigma_{10}\nabla\cdot(\sum_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) + R\vec{\Omega}_{10}\cdot\nabla(\sum_{\vec{\eta}} \dot{\sigma}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) +\\ & \textstyle +R(\sum_{\vec{\eta}'} \dot{\sigma}_{\vec{\eta}'} e^{i\vec{\eta}'\cdot\vec{\Theta}})\nabla\cdot(\sum_{\vec{\eta}-\vec{\eta}'} \vec{\Omega}_{\vec{\eta}-\vec{\eta}'} e^{i(\vec{\eta}-\vec{\eta}')\cdot\vec{\Theta}}) +\\ & \textstyle +R(\sum_{\vec{\eta}-\vec{\eta}'} \vec{\Omega}_{\vec{\eta}-\vec{\eta}'} e^{i(\vec{\eta}-\vec{\eta}')\cdot\vec{\Theta}})\cdot\nabla(\sum_{\vec{\eta}'} \dot{\sigma}_{\vec{\eta}'} e^{i\vec{\eta}'\cdot\vec{\Theta}})\end{aligned}$$

$$\begin{aligned}\eta \neq 0 \ : \ \dot{\sigma}_{\vec{\eta}} + 2\frac{V}{R}\sigma_{\vec{\eta}} + \sigma_0(i\vec{\eta}\cdot\vec{\Omega}_{\vec{\eta}}) + \sigma_{\vec{\eta}}(\vec{\Omega}_{10}\cdot i\vec{\eta})\\ +\sigma_0(i\vec{\eta}\cdot\vec{\Omega}_{\vec{\eta}}) + \sum_{\vec{\eta}'}(\vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\cdot i\vec{\eta})\sigma_{\vec{\eta}'}\end{aligned}$$

$$\dot{\sigma}_{\vec{\eta}} + \sigma_0(i\vec{\eta}\cdot\vec{\Omega}_{\vec{\eta}}) + 2\frac{V}{R}\sigma_{\vec{\eta}} + \sum_{\vec{\eta}'}(\vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\cdot i\vec{\eta})\sigma_{\vec{\eta}'} = 0 \tag{15}$$

Poisson equation

$$\Delta\Phi_1 = 4\pi G\sigma_1\delta(z),$$

We need:

$$\begin{array}{lll} z \neq 0 & \ldots & \Delta\Phi_1 = 0 \\ z = 0 & \ldots & \Phi_1 = \sum_{\vec{k}} \Phi_{1\vec{\eta}} e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{\eta}} \Phi_{1\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} \end{array}$$

ansatz: $\Phi_1 = \sum_{\vec{k}} \Phi_{1\vec{\eta}} e^{i\vec{k}\cdot\vec{r}-|\vec{k}|z}$
we integrate from $z = -\zeta$ to $z = +\zeta$, $\zeta \to 0$

$$\lim_{\zeta\to 0}\int_{-\zeta}^{\zeta}\frac{\partial^2\Phi_1}{\partial z^2}dz = 4\pi G\sigma_1\int_{-\zeta}^{\zeta}\delta(z)dz$$

$$\lim_{\zeta\to 0}\frac{\partial\Phi_1}{\partial z}\ |_{-\zeta}^{\zeta} = 4\pi G\sigma_1$$

$$-\sum_{\vec{k}} 2\vec{k}\Phi_{\vec{\eta}} e^{i\vec{k}\cdot\vec{r}} = 4\pi G\sum_{\vec{k}}\sigma_{\vec{\eta}} e^{i\vec{k}\cdot\vec{r}}$$

$$\Phi_{1\vec{\eta}} = -\frac{2\pi G\sigma_{\vec{\eta}}}{|\vec{k}|}$$

$$\Phi_1 = -2\pi G\sum_{\vec{k}}\frac{\sigma_{\vec{\eta}}}{|\vec{k}|}e^{i\vec{k}\cdot\vec{r}}$$

$$\nabla\Phi_1 = -2\pi G\sum_{\vec{k}}\sigma_{\vec{\eta}}\frac{i\vec{k}}{|\vec{k}|}e^{i\vec{k}\cdot\vec{r}} = -2\pi G\sum_{\vec{\eta}}\sigma_{\vec{\eta}}\frac{i\vec{\eta}}{|\vec{\eta}|}e^{i\vec{\eta}\cdot\vec{\Theta}} \tag{17}$$

Euler's equation

$$\begin{aligned} R(\dot{\vec{\Omega}}_{10} + \sum_{\vec{\eta}}\dot{\vec{\Omega}}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) + R^2(\vec{\Omega}_{10} + \sum_{\vec{\eta}}\vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})\cdot \\ \nabla(\vec{\Omega}_{10} + \sum_{\vec{\eta}}\vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) = -\frac{c^2}{\sigma_0}(\sigma_{10} + \sum_{\vec{\eta}}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) + \\ +\frac{c^2}{\sigma_0^2}(\sigma_{10} + \sum_{\vec{\eta}}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})\nabla(\sigma_{10} + \sum_{\vec{\eta}}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) + \\ +2\pi G\sum_{\vec{\eta}}\sigma_{\vec{\eta}}\frac{i\vec{\eta}}{|\vec{\eta}|}e^{i\vec{\eta}\cdot\vec{\Theta}} - 4V(\vec{\Omega}_{10} + \sum_{\vec{\eta}}\vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) + \\ +\frac{3V}{\sigma_0}(\vec{\Omega}_{10} + \sum_{\vec{\eta}}\vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}})(\sigma_{10} + \sum_{\vec{\eta}}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}}) \end{aligned}$$

$$\eta = 0 \ : \quad R\dot{\vec{\Omega}}_{10} + R^2 \textstyle\sum_{\vec{\eta}} (\vec{\Omega}_{\vec{\eta}} \cdot (-i\vec{k}))\vec{\Omega}_{-\vec{\eta}} = \frac{c^2}{\sigma_0^2} \textstyle\sum_{\vec{\eta}} \sigma_{\vec{\eta}} (-i\vec{k})\sigma_{-\vec{\eta}} - 4V\vec{\Omega}_{10} + \frac{3V}{\sigma_0}\vec{\Omega}_{10}\sigma_{10} + \frac{3V}{\sigma_0} \textstyle\sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}}\sigma_{-\vec{\eta}}$$

$$\begin{aligned} \eta \neq 0 \ : \quad & R \textstyle\sum_{\vec{\eta}} \dot{\vec{\Omega}}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + R^2 \textstyle\sum_{\vec{\eta}} (\vec{\Omega}_{10} \cdot i\vec{k})\vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \\ & + R^2 \textstyle\sum_{\vec{\eta}-\vec{\eta}'} \textstyle\sum_{\vec{\eta}'} (\vec{\Omega}_{\vec{\eta}-\vec{\eta}'} \cdot i\vec{k})\vec{\Omega}_{\vec{\eta}'} e^{i(\vec{\eta}-\vec{\eta}')\cdot\vec{\Theta}} e^{i\vec{\eta}'\cdot\vec{\Theta}} = \\ & = -\frac{c^2}{\sigma_0} \textstyle\sum_{\vec{\eta}} i\vec{k}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \frac{c^2}{\sigma_0^2}\sigma_{10} \textstyle\sum_{\vec{\eta}} i\vec{k}\sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \\ & + \frac{c^2}{\sigma_0^2} \textstyle\sum_{\vec{\eta}-\vec{\eta}'} \sigma_{\vec{\eta}-\vec{\eta}'} e^{i(\vec{\eta}-\vec{\eta}')\cdot\vec{\Theta}} \textstyle\sum_{\vec{\eta}'} i\vec{k}'\sigma_{\vec{\eta}'} e^{i\vec{\eta}'\cdot\vec{\Theta}} + \\ & + 2\pi G \textstyle\sum_{\vec{\eta}} \sigma_{\vec{\eta}} \frac{i\vec{\eta}}{|\vec{\eta}|} e^{i\vec{\eta}\cdot\vec{\Theta}} - 4V \textstyle\sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \\ & + \frac{3V}{\sigma_0}\vec{\Omega}_{10} \textstyle\sum_{\vec{\eta}} \sigma_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \frac{3V}{\sigma_0}\sigma_{10} \textstyle\sum_{\vec{\eta}} \vec{\Omega}_{\vec{\eta}} e^{i\vec{\eta}\cdot\vec{\Theta}} + \\ & + \frac{3V}{\sigma_0} \textstyle\sum_{\vec{\eta}-\vec{\eta}'} \vec{\Omega}_{\vec{\eta}-\vec{\eta}'} e^{i(\vec{\eta}-\vec{\eta}')\cdot\vec{\Theta}} \textstyle\sum_{\vec{\eta}'} \sigma_{\vec{\eta}'} e^{i\vec{\eta}'\cdot\vec{\Theta}} \end{aligned}$$

$$\begin{aligned} & R\dot{\vec{\Omega}}_{\vec{\eta}} + R(\vec{\Omega}_{10} \cdot i\vec{\eta})\vec{\Omega}_{\vec{\eta}} + R\sum_{\vec{\eta}'} (\vec{\Omega}_{\vec{\eta}-\vec{\eta}'} \cdot i\vec{\eta}')\vec{\Omega}_{\vec{\eta}'} = \\ & = -\frac{c^2}{\sigma_0} i\vec{k}\sigma_{\vec{\eta}} + -\frac{c^2}{\sigma_0^2}\sigma_{10} i\vec{k}\sigma_{\vec{\eta}} + \frac{c^2}{\sigma_0^2} \sum_{\vec{\eta}'} ik'\sigma_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'} \\ & \quad + 2\pi G \frac{i\vec{\eta}}{|\vec{\eta}|}\sigma_{\vec{\eta}} - 4V\vec{\Omega}_{\vec{\eta}} + \frac{3V}{\sigma_0}\vec{\Omega}_{10}\sigma_{\vec{\eta}} + \\ & \quad + \frac{3V}{\sigma_0}\vec{\Omega}_{\vec{\eta}}\sigma_{10} + \frac{3V}{\sigma_0} \sum_{\vec{\eta}'} \vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'}. \end{aligned}$$

$$\begin{aligned} R\dot{\vec{\Omega}}_{\vec{\eta}} + R\sum_{\vec{\eta}'} (\vec{\Omega}_{\vec{\eta}-\vec{\eta}'} \cdot i\vec{\eta}')\vec{\Omega}_{\vec{\eta}'} &= -\frac{c^2}{\sigma_0} i\vec{k}\sigma_{\vec{\eta}} + \frac{c^2}{\sigma_0^2} \sum_{\vec{\eta}'} ik'\sigma_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'} \\ & + 2\pi G \frac{i\vec{\eta}}{|\vec{\eta}|}\sigma_{\vec{\eta}} - 4V\vec{\Omega}_{\vec{\eta}} + \frac{3V}{\sigma_0} \sum_{\vec{\eta}'} \vec{\Omega}_{\vec{\eta}-\vec{\eta}'}\sigma_{\vec{\eta}'}. \end{aligned} \tag{16}$$

A.4 Linear analysis

$$\dot{\sigma}_{\vec{\eta}} + \sigma_0(i\vec{\eta} \cdot \vec{\Omega}_{\vec{\eta}}) + 2\frac{V}{R}\sigma_{\vec{\eta}} = 0 \tag{18}$$

$$R\dot{\vec{\Omega}}_{\vec{\eta}} = -\frac{c^2}{\sigma_0} i\vec{k}\sigma_{\vec{\eta}} + 2\pi G \frac{i\vec{\eta}}{|\vec{\eta}|}\sigma_{\vec{\eta}} - 4V\vec{\Omega}_{\vec{\eta}} \tag{19}$$

split $\vec{\Omega}_{\vec{\eta}}$ in two components:

$$\vec{\Omega}_{\vec{\eta}} = \Omega_{\vec{\eta}_{\|}} \frac{\vec{\eta}_{\|}}{\eta_{\|}} + \Omega_{\vec{\eta}_{\perp}} \frac{\vec{\eta}_{\perp}}{\eta_{\perp}} \tag{20}$$

$$\dot{\sigma}_{\vec{\eta}} = -2\frac{V}{R}\sigma_{\vec{\eta}} - i\eta\sigma_0\Omega_{\vec{\eta}_{\|}} \tag{21}$$

$$\dot{\Omega}_{\vec{\eta}_{\|}} = (i\frac{2\pi G}{R} - i\frac{c^2\eta}{\sigma_0 R^2})\sigma_{\vec{\eta}} - 4\frac{V}{R}\Omega_{\vec{\eta}_{\|}} \tag{22}$$

$$\dot{\Omega}_{\vec{\eta}_{\perp}} = -4\frac{V}{R}\Omega_{\vec{\eta}_{\perp}} \tag{23}$$

$$\sigma_{\vec{\eta}},\ \Omega_{\vec{\eta}} \sim e^{i\omega t} \tag{24}$$

find eigenvalues:

$$(i\omega + 4\frac{V}{R})[(i\omega + 2\frac{V}{R})(i\omega + 4\frac{V}{R}) - i\eta\sigma_0(i\frac{c^2\eta}{\sigma_0 R^2} - i\frac{2\pi G}{R})] = 0$$

$$\omega_{\vec{\eta}}^{(1,2)} = i\frac{3V}{R} \pm \sqrt{-\frac{V^2}{R^2} + \frac{\eta^2 c^2}{R^2} - \frac{2\pi G\sigma_0\eta}{R}} \tag{25}$$

$$\omega_{\vec{\eta}}^{(3)} = i4\frac{V}{R} \tag{26}$$

and eigenvectors:

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1,2)} = \begin{pmatrix} -i\eta\sigma_0 \\ i\omega^{(1,2)} + 2\frac{V}{R} \\ 0 \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{28}$$

Function $\omega^{(1,2)}(\eta)$ has maximum value at

$$\eta_{max} = \frac{\pi G\sigma_0 R}{c^2}\ , \tag{29}$$

and it is

$$\omega_{\vec{\eta},max}^{(1,2)} = i\frac{3V}{R} \pm \sqrt{-\frac{V^2}{R^2} - \frac{\pi^2 G^2\sigma_0^2}{c^2}}\ . \tag{30}$$

A.5 Non-linear analysis

$$\begin{pmatrix} \dot{\sigma}_{\vec{\eta}} \\ \dot{\Omega}_{\vec{\eta}_{\|}} \\ \dot{\Omega}_{\vec{\eta}_{\perp}} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix} + \mathcal{N} \tag{33}$$

$$\mathcal{N} = \begin{pmatrix} -i\sum_{\vec{\eta}'} \left(\Omega_{\vec{\eta}-\vec{\eta}'_{\|}} \frac{(\vec{\eta}-\vec{\eta}',\vec{\eta})}{|\vec{\eta}-\vec{\eta}'|} + \Omega_{\vec{\eta}-\vec{\eta}'_{\perp}} \frac{(\vec{\eta}-\vec{\eta}'_{\perp},\vec{\eta})}{|\vec{\eta}-\vec{\eta}'_{\perp}|} \right) \sigma_{\vec{\eta}'} \\ -i\sum_{\vec{\eta}'} \left(\Omega_{\vec{\eta}-\vec{\eta}'_{\|}} \frac{(\vec{\eta}-\vec{\eta}',\vec{\eta}')}{|\vec{\eta}-\vec{\eta}'|} + \Omega_{\vec{\eta}-\vec{\eta}'_{\perp}} \frac{(\vec{\eta}-\vec{\eta}'_{\perp},\vec{\eta}')}{|\vec{\eta}-\vec{\eta}'_{\perp}|} \right) \\ -i\sum_{\vec{\eta}'} \left(\Omega_{\vec{\eta}-\vec{\eta}'_{\|}} \frac{(\vec{\eta}-\vec{\eta}',\vec{\eta}')}{|\vec{\eta}-\vec{\eta}'|} + \Omega_{\vec{\eta}-\vec{\eta}'_{\perp}} \frac{(\vec{\eta}-\vec{\eta}'_{\perp},\vec{\eta}')}{|\vec{\eta}-\vec{\eta}'_{\perp}|} \right) \\ +0 \\ \left(\Omega_{\vec{\eta}'_{\|}} \frac{(\vec{\eta}',\vec{\eta})}{|\vec{\eta}'||\vec{\eta}|} + \Omega_{\vec{\eta}'_{\perp}} \frac{(\vec{\eta}'_{\perp},\vec{\eta})}{|\vec{\eta}'_{\perp}||\vec{\eta}|} \right) + i\frac{c^2}{\sigma_0^2 R^2} \sum_{\vec{\eta}'} \sigma_{\vec{\eta}'-\vec{\eta}} \sigma_{\vec{\eta}'} \frac{\vec{\eta}',\vec{\eta}}{|\vec{\eta}|} \\ \left(\Omega_{\vec{\eta}'_{\|}} \frac{(\vec{\eta}',\vec{\eta}_{\perp})}{|\vec{\eta}'||\vec{\eta}_{\perp}|} + \Omega_{\vec{\eta}'_{\perp}} \frac{(\vec{\eta}'_{\perp},\vec{\eta}_{\perp})}{|\vec{\eta}'_{\perp}||\vec{\eta}_{\perp}|} \right) + i\frac{c^2}{\sigma_0^2 R^2} \sum_{\vec{\eta}'} \sigma_{\vec{\eta}'-\vec{\eta}} \sigma_{\vec{\eta}'} \frac{\vec{\eta}',\vec{\eta}_{\perp}}{|\vec{\eta}_{\perp}|} \\ +0 \\ +\frac{3V}{\sigma_0 R} \sum_{\vec{\eta}'} \left(\Omega_{\vec{\eta}'-\vec{\eta}_{\|}} \frac{(\vec{\eta}-\vec{\eta}',\vec{\eta})}{|\vec{\eta}-\vec{\eta}'||\vec{\eta}|} + \Omega_{\vec{\eta}'-\vec{\eta}_{\perp}} \frac{(\vec{\eta}-\vec{\eta}'_{\perp},\vec{\eta})}{|\vec{\eta}-\vec{\eta}'_{\perp}||\vec{\eta}|} \right) \sigma_{\vec{\eta}'} \\ +\frac{3V}{\sigma_0 R} \sum_{\vec{\eta}'} \left(\Omega_{\vec{\eta}'-\vec{\eta}_{\|}} \frac{(\vec{\eta}-\vec{\eta}',\vec{\eta}_{\perp})}{|\vec{\eta}-\vec{\eta}'||\vec{\eta}_{\perp}|} + \Omega_{\vec{\eta}'-\vec{\eta}_{\perp}} \frac{(\vec{\eta}-\vec{\eta}'_{\perp},\vec{\eta}_{\perp})}{|\vec{\eta}-\vec{\eta}'_{\perp}||\vec{\eta}_{\perp}|} \right) \sigma_{\vec{\eta}'} \end{pmatrix}$$

search for solution in form

$$\begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix} = \psi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} + \xi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} + \phi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} \tag{34}$$

$$\dot{\psi}_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} \psi(t) + \dot{\xi}_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} +$$

$$+\frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} \xi(t) + \dot{\phi}_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} + \frac{\partial}{\partial t} \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} \phi(t) =$$

$$= i\omega^{(1)} \psi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(1)} + i\omega^{(2)} \xi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(2)} +$$

$$+i\omega^{(3)} \phi_{\vec{\eta}}(t) \begin{pmatrix} \sigma_{\vec{\eta}} \\ \Omega_{\vec{\eta}_{\|}} \\ \Omega_{\vec{\eta}_{\perp}} \end{pmatrix}^{(3)} + \mathcal{N}$$

we find orthonormal vectors $(\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(1,2,3)}$ in order that

$$(\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(i)} \begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(i)} = 1, \qquad (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(i)} \begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(j\neq i)} = 0$$

$$\begin{aligned}
(\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(1)} &= \left(\tfrac{i\omega^{(2)}+2\frac{V}{R}}{\eta\sigma_0(\omega^{(1)}-\omega(2))}, \tfrac{i}{(\omega^{(1)}-\omega(2))}, 0\right) \\
(\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(2)} &= \left(\tfrac{i\omega^{(1)}+2\frac{V}{R}}{\eta\sigma_0(\omega^{(2)}-\omega(1))}, \tfrac{i}{(\omega^{(2)}-\omega(1))}, 0\right) \\
(\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(3)} &= (0,0,1)
\end{aligned} \tag{35}$$

We multiply equation (33) by orthonormal vectors (35) and get a set of equation

$$\dot\psi_{\vec\eta} = i\omega^{(1)}\psi_{\vec\eta} + \frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(1)} \cdot (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(1)}\psi_{\vec\eta} +$$
$$+\frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(2)} \cdot (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(1)}\xi_{\vec\eta} + (\mathcal{N}, (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(1)}) \tag{36}$$

$$\dot\xi_{\vec\eta} = i\omega^{(2)}\xi_{\vec\eta} + \frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(2)} \cdot (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(2)}\xi_{\vec\eta} +$$
$$+\frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(1)} \cdot (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(2)}\psi_{\vec\eta} + (\mathcal{N}, (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(2)}) \tag{37}$$

$$\dot\phi_{\vec\eta} = i\omega^{(3)}\phi_{\vec\eta} + \frac{\partial}{\partial t}\begin{pmatrix} \sigma_{\vec\eta} \\ \Omega_{\vec\eta_\parallel} \\ \Omega_{\vec\eta_\perp} \end{pmatrix}^{(3)} \cdot (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(3)}\phi_{\vec\eta} + (\mathcal{N}, (\sigma_{\vec\eta}, \Omega_{\vec\eta_\parallel}, \Omega_{\vec\eta_\perp})^{(3)}) \tag{38}$$

$$\begin{aligned}
\sigma_{\vec\eta} &= -i\eta\sigma_0(\psi_{\vec\eta} + \xi_{\vec\eta}) \\
\Omega_{\vec\eta_\parallel} &= 2\frac{V}{R}(\psi_{\vec\eta} + \xi_{\vec\eta}) + i(\omega^{(1)}\psi_{\vec\eta} + \omega^{(2)}\xi_{\vec\eta}) \\
\Omega_{\vec\eta_\perp} &= \phi_{\vec\eta}
\end{aligned}$$

$$\begin{aligned}
\dot{\xi}_{\vec{\eta}} &= i\omega^{(2)}\xi_{\vec{\eta}} - \frac{i(\dot{\eta}\sigma_0 + \eta\dot{\sigma}_0)(i\omega^{(1)} + 2\frac{V}{R}) + i\eta\sigma_0(i\dot{\omega}^{(2)} + 2\frac{\dot{V}R - V^2}{R^2})}{\eta\sigma_0(\omega^{(1)} - \omega^{(2)})}\xi_{\vec{\eta}} \\
& - \frac{i(\dot{\eta}\sigma_0 + \eta\dot{\sigma}_0)(i\omega^{(1)} + 2\frac{V}{R}) + i\eta\sigma_0(i\dot{\omega}^{(2)} + 2\frac{\dot{V}R - V^2}{R^2})}{\eta\sigma_0(\omega^{(2)} - \omega^{(2)})}\psi_{\vec{\eta}} \\
& + \frac{\omega^{(1)} - i2\frac{V}{R}}{\eta\sigma_0\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right) \right.\right. \\
& \left.\left. + i\left(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'}\right)\right] \frac{(\vec{\eta}-\vec{\eta}', \vec{\eta})}{|\vec{\eta}-\vec{\eta}'|} + \phi_{\vec{\eta}-\vec{\eta}'}\frac{(\vec{\eta}-\vec{\eta}'_{\perp}, \vec{\eta})}{|\vec{\eta}-\vec{\eta}'_{\perp}|} \right\} \\
& \left[-i\eta'\sigma_0\left(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'}\right)\right] + \frac{1}{\omega^{(1)} - \omega^{(2)}} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right) \right.\right. \\
& \left.\left. + i(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'})\right] \frac{(\vec{\eta}-\vec{\eta}', \vec{\eta}')}{|\vec{\eta}-\vec{\eta}'|} + \phi_{\vec{\eta}-\vec{\eta}'}\frac{(\vec{\eta}-\vec{\eta}'_{\perp}, \vec{\eta}')}{|\vec{\eta}-\vec{\eta}'_{\perp}|} \right\} \\
& \left\{ \left[2\frac{V}{R}\left(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'}\right) + i\left(\omega^{(1)}\psi_{\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}'}\right)\right] \frac{(\vec{\eta}', \vec{\eta})}{|\vec{\eta}'|\,|\vec{\eta}|} + \phi_{\vec{\eta}'}\frac{(\vec{\eta}'_{\perp}, \vec{\eta}')}{|\vec{\eta}'_{\perp}|\,|\vec{\eta}|} \right\} \\
& - \frac{c^2}{\sigma_0^2 R^2\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left[-i\left(\eta - \eta'\right)\sigma_0\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right)\right] \qquad (39) \\
& \left[-i\eta'\sigma_0(\psi_{\vec{\eta}'} + \xi_{\vec{\eta}'})\right] \frac{(\vec{\eta}', \vec{\eta})}{|\vec{\eta}|} + \frac{i3V}{R\sigma_0\left(\omega^{(1)} - \omega^{(2)}\right)} \sum_{\vec{\eta}'} \left\{ \left[2\frac{V}{R} \right.\right. \\
& \left.\left(\psi_{\vec{\eta}-\vec{\eta}'} + \xi_{\vec{\eta}-\vec{\eta}'}\right) + i(\omega^{(1)}\psi_{\vec{\eta}-\vec{\eta}'} + \omega^{(2)}\xi_{\vec{\eta}-\vec{\eta}'})\right] \frac{(\vec{\eta}-\vec{\eta}', \vec{\eta})}{|\vec{\eta}-\vec{\eta}'|\,|\vec{\eta}|} \\
& \left. + \phi_{\vec{\eta}-\vec{\eta}'}\frac{(\vec{\eta}-\vec{\eta}'_{\perp}, \vec{\eta})}{|\vec{\eta}-\vec{\eta}'_{\perp}|\,|\vec{\eta}|} \right\} \left[-i\eta'\sigma_0(\mu_{\vec{\eta}'} + \nu_{\vec{\eta}'})\right]
\end{aligned}$$

From geometry:

$$\begin{array}{ll} (\vec{\eta}, \vec{\eta}'_{\pm}) = \frac{1}{2}\eta_{max}^2 & (\vec{\eta}, \vec{\eta} - \vec{\eta}'_{\pm}) = \frac{1}{2}\eta_{max}^2 \\ (\vec{\eta}'_{+}, \vec{\eta} - \vec{\eta}'_{+}) = -\frac{1}{2}\eta_{max}^2 & (\vec{\eta}'_{-}, \vec{\eta} - \vec{\eta}'_{-}) = -\frac{1}{2}\eta_{max}^2 \end{array} . \qquad (40)$$

We count sums over η_+ and η_-, insert $\omega^{(2)}$ and scalar products and get:

$$\begin{aligned}
\dot{\xi} &= i\omega^{(2)}\xi + \alpha\xi - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi_+\xi_- \\
\dot{\xi}_+ &= i\omega^{(2)}\xi_+ + \alpha\xi_+ - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi\xi_-^* \qquad (41) \\
\dot{\xi}_- &= i\omega^{(2)}\xi_- + \alpha\xi_- - \frac{1}{4}\frac{\pi G\sigma_0\left(8V^2c^2 - 3\pi^2G^2\sigma_0^2R^2 + i8c^2RV\Sigma\right)}{Rc^4\Sigma}\xi\xi_+^* ,
\end{aligned}$$

where

$$\Sigma = \sqrt{-\frac{V^2c^2 + \pi^2 G^2 \sigma_0^2 R^2}{R^2c^2}} \tag{42}$$

and

$$\begin{aligned}\alpha \quad = \quad & \frac{-i2\dot{\sigma}_0 R^2 V c^2 \Sigma - 2\dot{\sigma}_0 R^3 c^2 \Sigma^2 - \sigma_0 R^2 V c^2 \Sigma^2 - i\sigma_0 R^2 \dot{V} c^2 \Sigma}{2\sigma_0 R^3 c^2 \Sigma} + \\ & + \frac{\sigma_0 R V \dot{V} c^2 + \sigma_0 V^3 c^2 + \sigma_0^2 \dot{\sigma}_0 \pi^2 G^2 R^3}{2\sigma_0 R^3 c^2 \Sigma} \end{aligned} \tag{43}$$