HD Simulations of Expanding Shells
The Gravitational Instability

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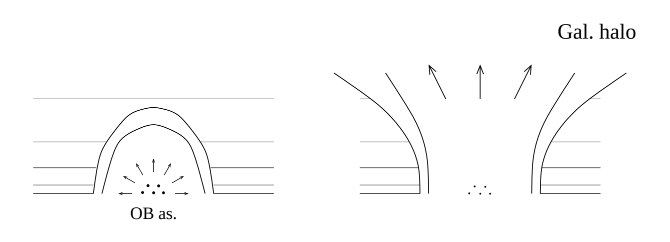
1. The grav. instability of expanding shells
   the linear analysis and the mass spectrum
2. The ZEUS code
3. The multigrid Poisson solver
4. Simulations of the Expanding Shell

HI shells

- **basic characteristics:**
  $R \sim 10\ pc - 2$ kpc, $E \sim 10^{51} - 10^{54}$ erg, $v_{exp} \sim 5 - 30$ km/s
- **Observations:**
  MW: Heiles (1979); recent obs.: McClure-Griffiths et al. (2002)
  external galaxies: LMC (Kim, 1998), SMC (Stanimirovic, 1999)
  M31, M33, M101, HoII, IC10
- **origin - SN explosion, OB association, group of OB as., encounter with HVC or a dwarf galaxy, GRB**
- **Why are HI shells interesting?**
  - **Propagating Star Formation:**
    OB as. $\rightarrow$ exp. HI shell $\rightarrow$ fragments $\rightarrow$ new stars
  - **Disc Halo Connection**



Gravitational instability - the linear analysis

Elmegreen (1994)

Assume a cold and thin shell of the radius r expanding with the velocity v into the medium of the density ρ_0. HD equations and the Poisson equation on the shell surface:

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \Omega) = v \rho_0,$$

$$\frac{\partial}{\partial t} (\Sigma \Omega) + (\Omega \cdot \nabla) \Sigma \Omega = -\Sigma \nabla \Phi - c_{sh}^2 \nabla \Sigma$$

$$\Delta \Phi = 4\pi G \Sigma \delta (R - r)$$

Σ	–	surface density of the shell
Ω	–	2D angular velocity of the surface flows
c_{sh}	–	sound speed in the shell
Φ	–	gravitational potential

→ **linearization, stability analysis of the Fourier modes** $(\Sigma_{\text{pert}} \sim \exp(\omega t))$ (parametrised by the dimensionless wavenumber $\eta = kr$):

$$\omega(\eta) = -\frac{3v}{r} + \sqrt{\frac{v^2}{r^2} + \frac{2\pi G \Sigma_0 \eta}{r} - \frac{c^2 \eta^2}{r^2}}$$

Mass spectrum of fragments

- can be derived from the dispersion rel. $\omega(\eta)$ (from the linear analysis)
- the fragmentation integral:

$$I_f(\eta, t) = \int_{t_B}^{t} \omega(\eta, t')dt' \ ,$$

where t_B is the time when the instability begins

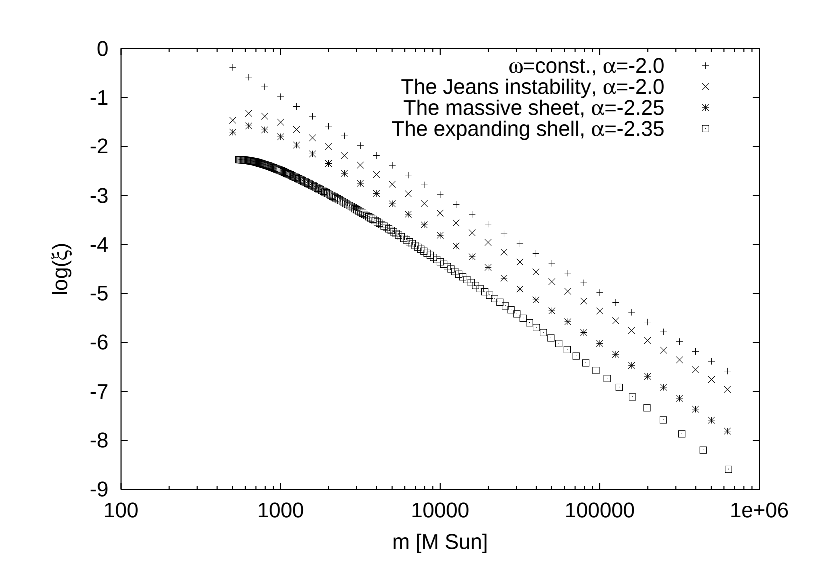
- the mass spectrum:

$$dN \sim I_f(\eta, t) \times \frac{r^2}{\lambda^2} d\eta$$

$$\rightarrow dN \sim m^{-\alpha} dm \ ,$$

where α is mass spectrum index

expanding shell yields the mass spectrum index $\alpha = 2.35$ at high mass end (very close to the stellar IMF) independently on the parameters of the shell and on the stage of its evolution, conf. to other physical situations



The ZEUS hydrocode (Stone & Norman, 1992)

- 3D second-order accuracy hydrodynamic code
- written in a covariant formalism $\Rightarrow$ works in arbitrary orthogonal coordinates
- based on method of finite differences and finite volumes (simple, fast and robust)
- exists also in parallel version (but we don't use it yet)
- solves the set of equations:

The hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \rho \vec{v} = -\nabla p - \rho \nabla \Phi$$

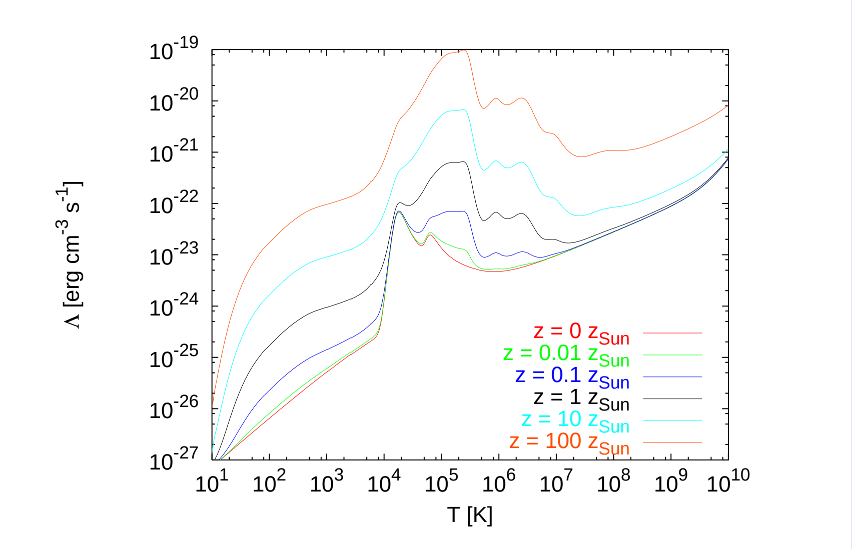
$$\rho \frac{\partial}{\partial t}\left(\frac{e}{\rho}\right) + \rho \vec{v} \cdot \nabla \left(\frac{e}{\rho}\right) = -p \nabla \cdot \vec{v}$$

+ the equation of state:

e.g. $p = (\gamma - 1)e$

ZEUS: new parts of code

- The multigrid Poisson solver
- The radiative cooling: $\frac{dE}{dt} = -\rho^2 \Lambda(T, z)$
  (The cooling function was kindly provided by Prof. Różyczka)



- The additional pressure: $p_{\rm add} = \rho c_{\rm add}^2$
  (in reality turbulent pressure; alternative: magnetic pressure)
- several bugs fixed (total energy advection, . . .)
  (in colaboration with A. Gawryszczak)

Poisson equation - discretisation

Poisson solver: $\rho(x_1, x_2, x_3) \rightarrow \Phi(x_1, x_2, x_3)$

Poisson equation $\nabla^2\Phi = 4\pi G\rho$ **must be fulfilled**

Discretization of the PE in general coordinates:

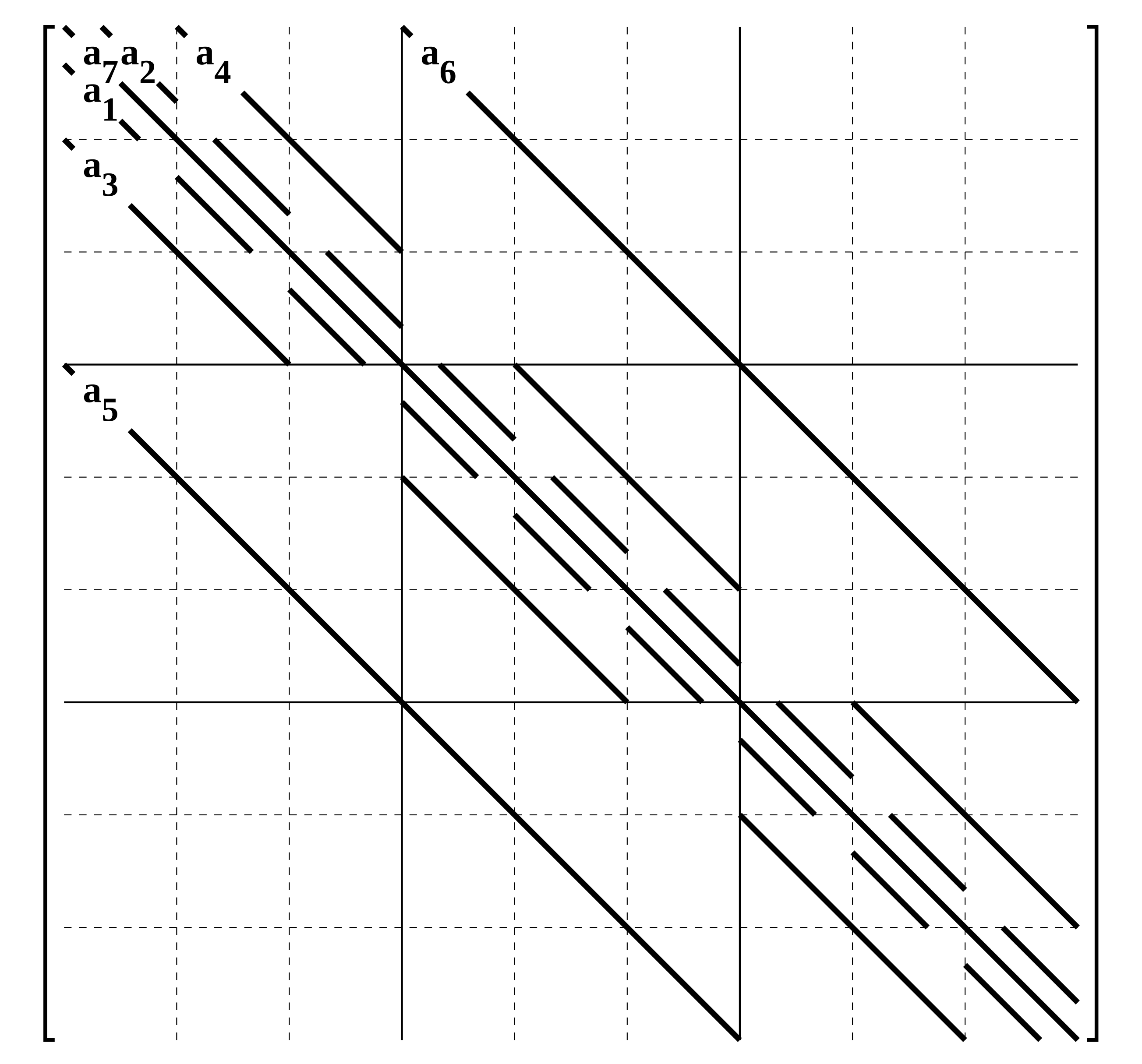
$$
\begin{aligned}
\nabla^2\Phi &= a_1\Phi_{i-1,j,k} + a_2\Phi_{i+1,j,k} + a_3\Phi_{i,j-1,k} \\
&+ a_4\Phi_{i,j+1,k} + a_5\Phi_{i,j,k-1} + a_6\Phi_{i+1,j,k+1} \\
&+ a_7\Phi_{i,j,k} = 4\pi G\rho_{i,j,k}
\end{aligned}
$$

this algebraic equation must be fulfilled in each cell

$\Rightarrow$ **we have the set of** $L \times M \times N$ **equations for a simulation on the** $L \times M \times N$ **grid**

$$A\mathbf{u} = \mathbf{f}$$

Poisson solver (Poisson equation matrix)



Poisson equation: methods of solving

1. **Direct methods:**
   - **Gaussian elimination -** ineffective for sparse matrices
   - **Spectral methods (FFT, spherical harmonics) -** quick, but not general
2. **Iterative methods:**
   - **standard relaxation (Jacobi, Gauss-Seidel) -** problem with smooth modes
   - **multigrid -** general, effective, not easy to code

Iterative methods

Set of linear equations:

$$\boxed{A\mathbf{u} = \mathbf{f}} \quad (*)$$

The residual equation:

$$\boxed{A\mathbf{e} = \mathbf{r}} \quad (**)$$

$\mathbf{u}$	...	the exact solution of (*)
$\mathbf{v}$	...	approximation to the exact solution
$\mathbf{e} = \mathbf{v} - \mathbf{u}$	...	the error
$\mathbf{r} = \mathbf{f} - A\mathbf{v}$	...	the residual

Poisson equation: Iterative methods

General iteration scheme:

$$A = D - L - U \;\rightarrow\; (D - L - U)\mathbf{u} = \mathbf{f}$$

$$D\mathbf{u} = (L + U)\mathbf{u} + f \;\rightarrow\; \mathbf{u} = \underbrace{D^{-1}(L + U)}_{R}\mathbf{u} + D^{-1}\mathbf{f}$$

$$\mathbf{v}^{(1)} = R\mathbf{v}^{(0)} + D^{-1}\mathbf{f}$$

Convergence rate: eigenvalues of R

Example: Jacobi method in 1D

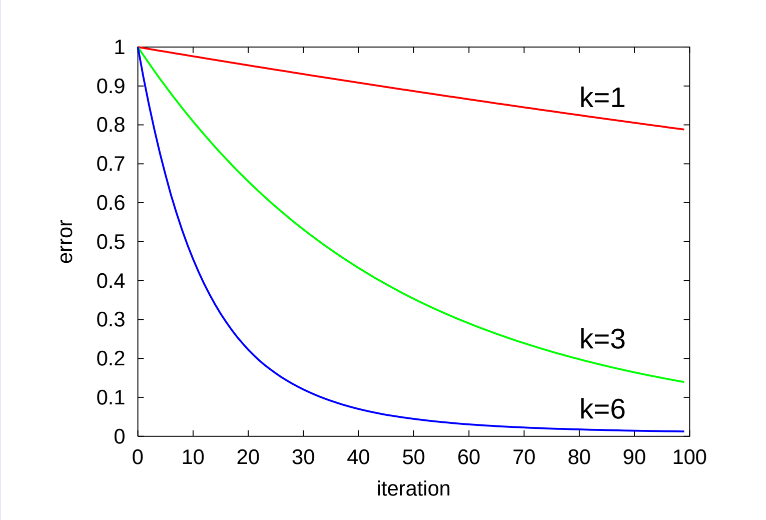
$$-\mathbf{u}_{xx} = \mathbf{f}$$

discretization: $-u_{j-1} + 2u_j - u_{j+1} = h^2 f_j$

new iteration:

$$v_j^{(1)} = \frac{1}{2}(v_{j-1}^{(0)} + v_{j+1}^{(0)} + h^2 f_j)$$

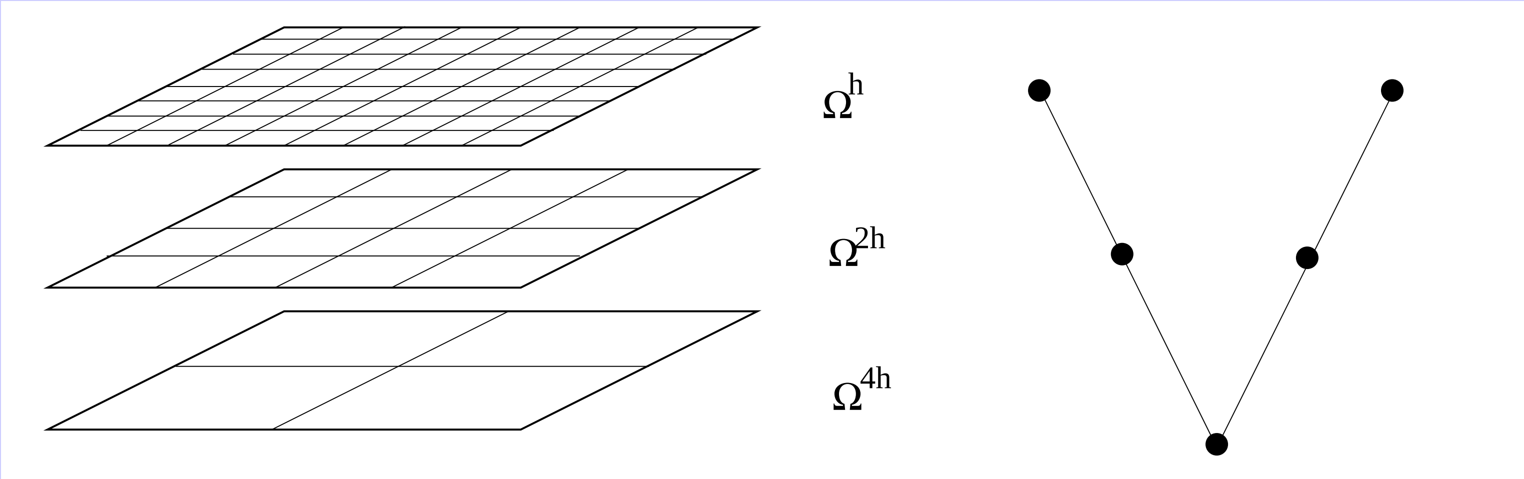
Convergence rate of various Fourier modes of the error:



Problem: standard relaxation methods are very ineffective in reducing the smooth modes of the error.

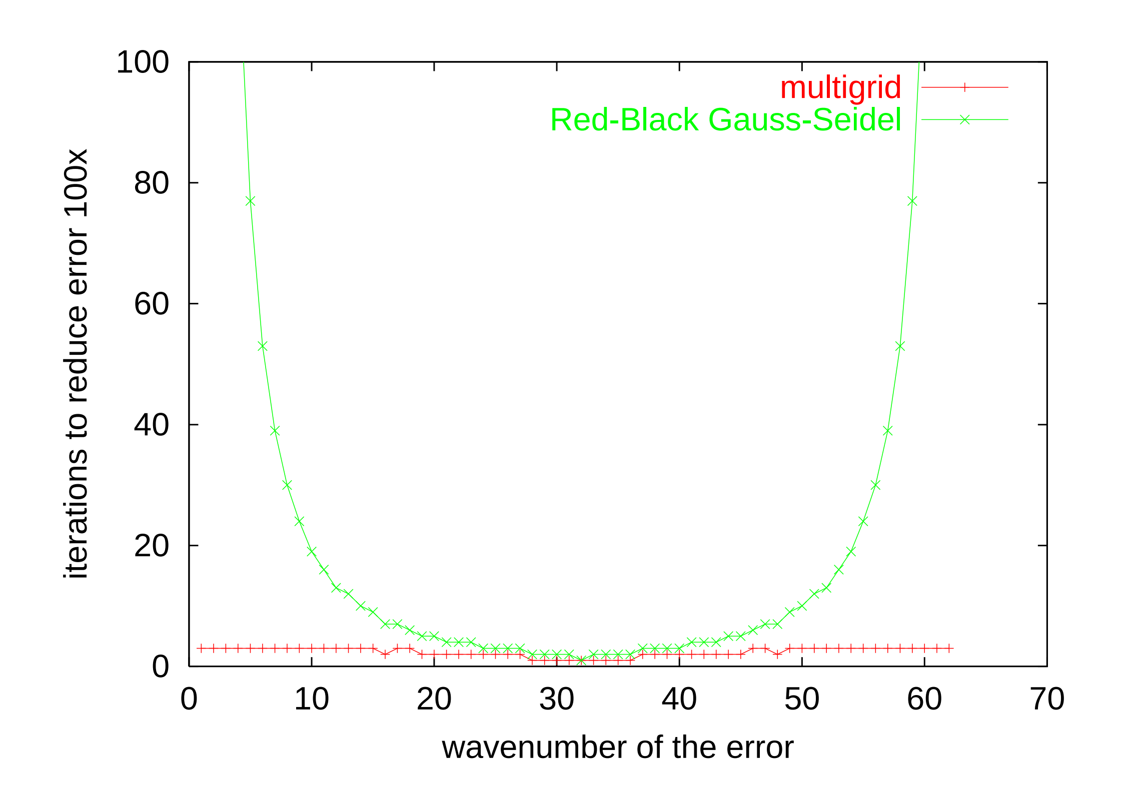
Poisson solver: Multigrid method

- basic idea: long wavelength modes look more oscillatory on a coarser grid
- the V-cycle scheme:
  1. relax on the fine grid
  2. restrict the problem to the coarser grid and do a relaxation
  3. . . .
  4. solve the set of equations on the coarsest grid (exactly)
  5. interpolate to a finer grid and do a relaxation
  6. . . .
  7. final relaxation of the finest grid



Multigrid vs. standard relaxation

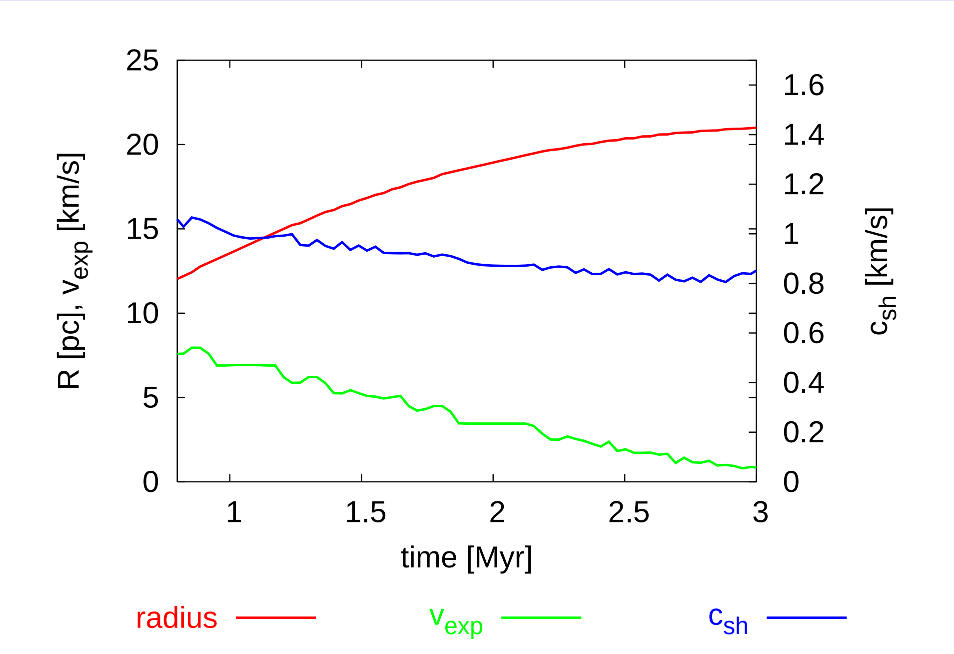
relaxation on 64x64x64 grid – multigrid vs. Red-Black Gauss-Seidel method
number of iterations to reduce error 100x
(RGBS needs more than 1000 iterations to reduce error with $k = 1$)

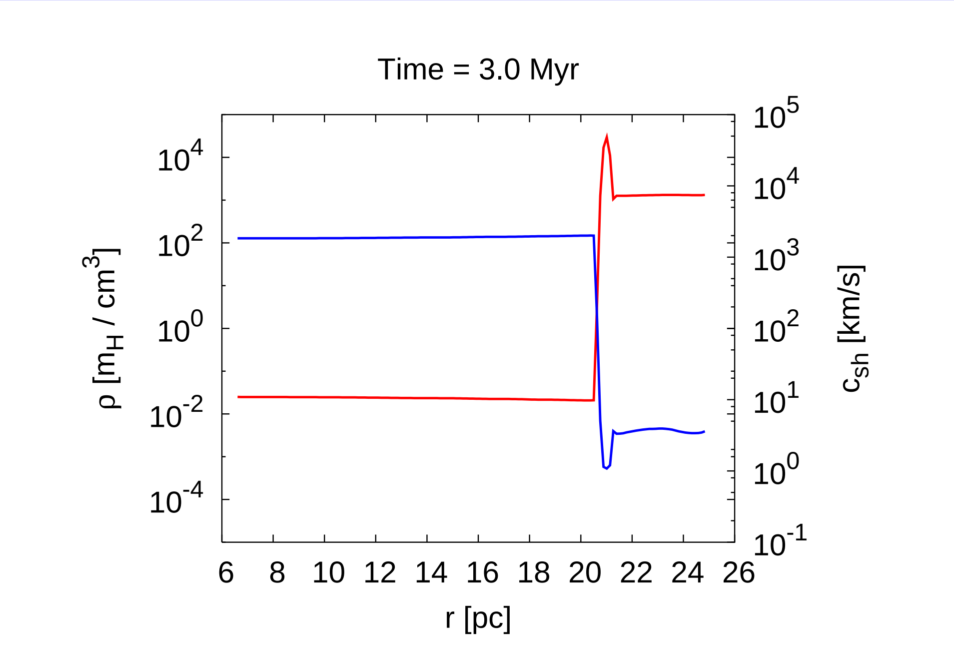


Simulations of the Expanding Shell

Model of the shell:
Insert the energy at a rate L into the homogeneous medium of the density ρ_0 and the temperature T_0. The spherical expanding shock is created.

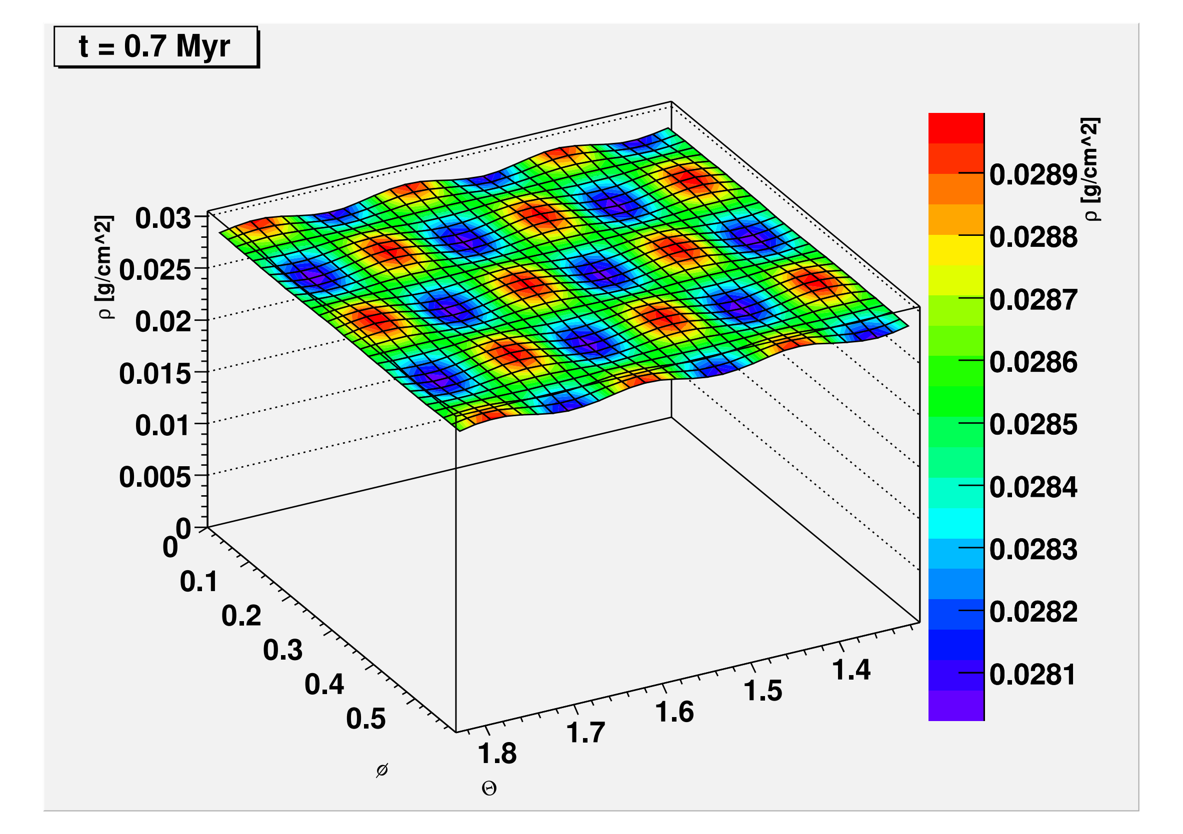
Parameters: $L = 5 \times 10^{51}\mathrm{erg/Myr}$, $n_0 = 10^3\mathrm{cm}^{-3}$, $\mu = 1.3$, $\rho_0 = n_0\mu$, $T_0 = 10^3\mathrm{K}$





Initial perturbations

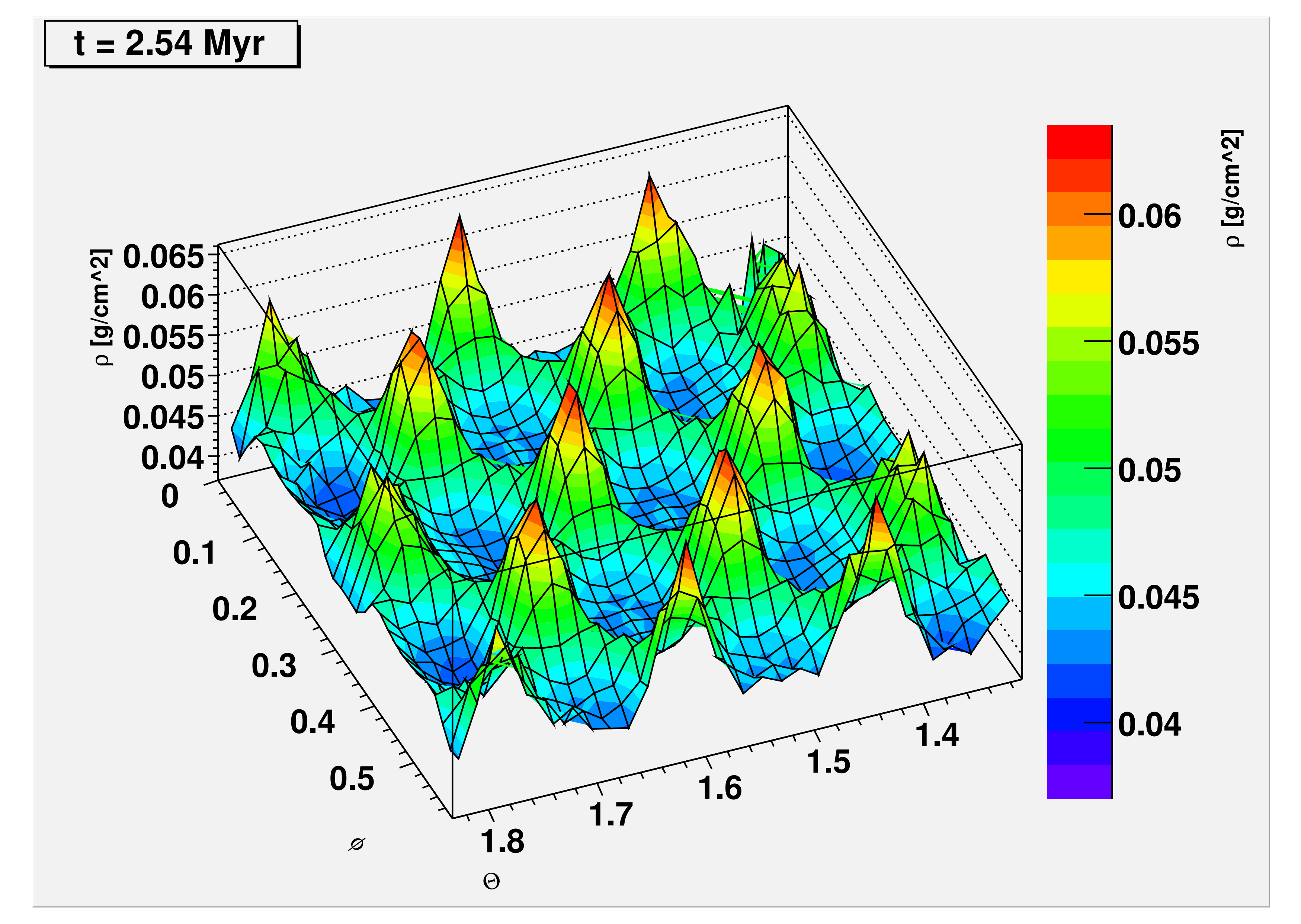
The density is perturbed in the shell when the grav. istability should begin.



Parameters:

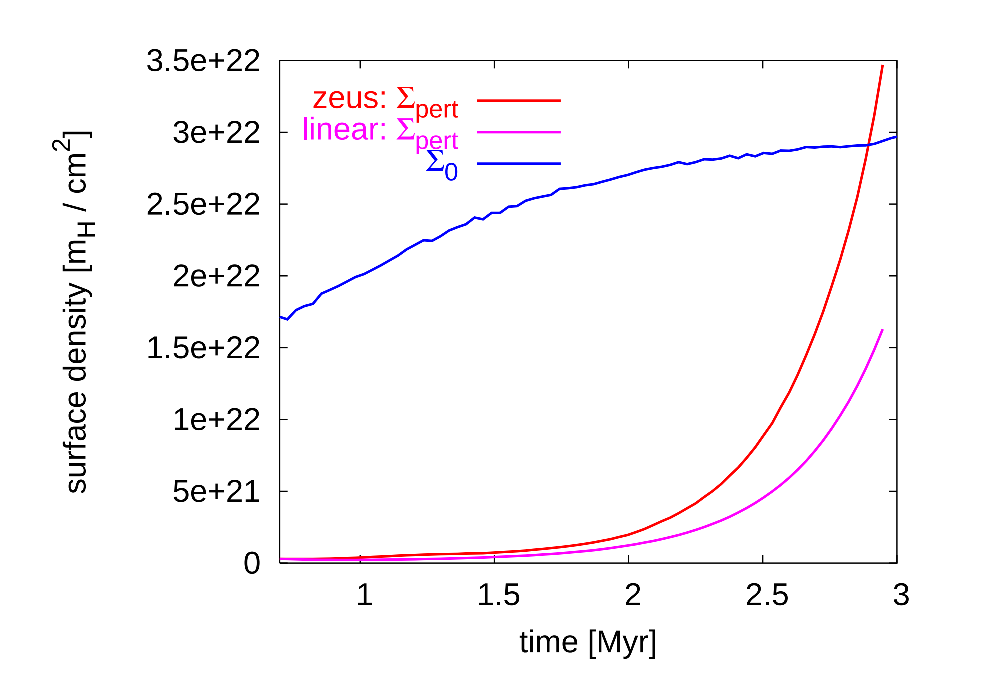
axial distr.: $\Lambda_l^m(\theta, \phi)$ **l=42, m=21**
(the most unstable mode)
amplitude: 2%
computed area: $30^o x 30^o$
grid: 150x32x32

Fragments on the surface of the shell



The HD simulation vs. the linear analysis

The perturbation growth rate in the HD simulation comparing to the ω from the linear analysis. $\omega(R, v_{exp}, \Sigma_0, c_{sh})$; we take the average values of R, v_{exp}, Σ_0, c_{sh} from the HD simulation.



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