Dynamics in the vicinity of the circular trajectory:

cylindrical coordinates : $(R, \vartheta, z, \dot{R}, \dot{\vartheta}, \dot{z})$

Assumptions :

1. $\Phi \neq \Phi(t)$ - quasiequilibrium
2. $\Phi = \Phi(\vartheta)$ - axial symmetry
3. $\Phi(R, z) = \Phi(R, -z)$ - plane symmetry
4. $\Phi(R, z) = \Phi_1(R) + \Phi_2(z)$ - pot. is separable

Integrals of motion :

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\vartheta}^2 + \dot{z}^2\right) + \Phi(R, z)$$
$$h = R^2\dot{\vartheta}$$

Equations of motion :

from : $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \emptyset$, $L = \frac{1}{2}\left(\dot{R}^2 + R\dot{\vartheta}^2 + \dot{z}^2\right) - \Phi(R, z)$

$$\ddot{R} = R\dot{\vartheta}^2 - \frac{\partial \Phi}{\partial R}$$

$$\frac{d}{dt}\left(R^2\dot{\vartheta}\right) = -\frac{\partial \Phi}{\partial \vartheta} = \emptyset$$

$$\ddot{z} = -\frac{\partial \Phi}{\partial z}$$

Circular orbit :

$$\ddot{R}_0 = 0 \quad \Rightarrow \quad R_0\dot{\vartheta}_0^2 = \left(\frac{\partial \Phi}{\partial R}\right)_0 = \frac{\theta_0^2}{R_0}$$

$$h_0^2 = R_0^2\theta_0^2 = R_0^3\left(\frac{\partial \Phi}{\partial R}\right)_0 \quad \Rightarrow \quad h_0 = R_0^2\dot{\vartheta}_0$$

Nearly circular orbit:

$$R = R_0 + R_1 \quad , \quad R_1 \ll R_0$$

$$\ddot{R} = (R_0 + R_1)\dot{\vartheta}^2 - \frac{\partial \Phi}{\partial R}$$

$$\ddot{R}_1 = (R_0+R_1)\dot{\vartheta}^2 - \left(\frac{\partial\Phi}{\partial R}\right)_0 - R_1\left(\frac{\partial^2\Phi}{\partial R^2}\right)_0$$

$$\ddot{R}_1 = \frac{h^2}{(R_0+R_1)^3} - \left(\frac{\partial\Phi}{\partial R}\right)_0 - R_1\left(\frac{\partial^2\Phi}{\partial R^2}\right)_0$$

$$\ddot{R}_1 = \frac{h^2}{R_0^3}\left(\cancel{1} - 3\frac{R_1}{R_0}\right) - \cancel{\left(\frac{\partial\Phi}{\partial R}\right)_0} - R_1\left(\frac{\partial^2\Phi}{\partial R^2}\right) \qquad h_0^2 = R_0^3\left(\frac{\partial\Phi}{\partial R}\right)_0$$

$$\ddot{R}_1 = -3\frac{h_0^2}{R_0^3}\frac{R_1}{R_0} - R_1\left(\frac{\partial^2\Phi}{\partial R^2}\right)_0$$

taking $h = h_0$ - on this circular orbit; other h belongs to different circ. orbit

$$\ddot{R}_1 = -\underbrace{\left[\frac{3}{R_0}\left(\frac{\partial\Phi}{\partial R}\right)_0 + \left(\frac{\partial^2\Phi}{\partial R^2}\right)_0\right]}_{\varkappa^2 \text{ - epicyclic frequency}} R_1$$

$$\boxed{\ddot{R}_1 = -\varkappa^2 R_1}$$

stability for $\varkappa^2 > 0$:

$$\varkappa^2 R_0^3 = \left[\frac{\partial}{\partial R}\left(R^3\frac{\partial\Phi}{\partial R}\right)\right]_0$$

$\frac{\partial\Phi}{\partial R} \sim R^\alpha$, $\alpha > -2$ always $\Rightarrow$ orbit is always stable

Using Oort's consts :

$$F_{R,0} = \left(\frac{\partial\Phi}{\partial R}\right)_0 = \frac{\Theta_0^2}{R_0} \quad , \quad \left(\frac{\partial F_R}{\partial R}\right)_0 = \left(\frac{\partial^2\Phi}{\partial R^2}\right)_0 = -\frac{\Theta_0^2}{R_0^2} + 2\frac{\Theta_0}{R_0}\left(\frac{d\Theta}{dR}\right)_0$$

$$\varkappa^2 = \frac{3}{R_0}\left(\frac{\partial\Phi}{\partial R}\right)_0 + \left(\frac{\partial^2\Phi}{\partial R^2}\right)_0 = \frac{3}{R_0}\frac{\Theta^2}{R_0} - \frac{\Theta_0^2}{R_0^2} + 2\frac{\Theta_0}{R_0}\left(\frac{d\Theta}{dR}\right)_0$$

$$\varkappa^2 = 2\underbrace{\frac{\Theta_0}{R_0}}_{(A-B)}\underbrace{\left[\frac{\Theta_0}{R_0} + \left(\frac{d\Theta}{dR}\right)_0\right]}_{2B} = 4(A-B)B$$

$\varkappa^2 \doteq 1300$, $\varkappa \doteq 36\ \mathrm{km\,s^{-1}\,kpc^{-1}}$, $T_{ep} = \frac{2\pi}{\varkappa} \doteq 170\,\mathrm{Myr}$