Treating stars as a fluid

- Collisionless system
  - stars in a galaxy or star cluster (with some care)
  - does not mean that stars do not collide physically
    -> it is much stronger assumption
  - grav. potential can be split into two parts

$$\Phi(\vec{x},t) = \Phi_0(\vec{x},\cancel{t}) + \Phi_1(\vec{x},t)$$

equilibrium

background pot. smooth, contrib from distant stars

fluctuations due to close encounters, leading to relaxation

  - collisionless $\Longleftrightarrow$ $\Phi_1(\vec{x},t) = \emptyset$
  - systems with age (or period of our interest) ~~can be~~ smaller than t_{relax} can be treated as collisionlell

- Unifying terminology
  - assume a scalar field $f(\vec{x})$

    + gradient : ∇f

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \quad \text{in Cartesian coords}$$

$$\nabla f = \left(\frac{\partial f}{\partial R}, \frac{1}{R}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z}\right) \quad \text{in cylindrical coords}$$

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \vartheta}, \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \phi}\right) \quad \text{in spherical coords}$$

    + divergence of vector field $\vec{F}(\vec{x})$: $\nabla \cdot \vec{F}$

$$\nabla \cdot \vec{F}(\vec{x}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i} \equiv \frac{\partial F_i}{\partial x_i}$$

+ divergence theorem

$$\int_V \nabla\cdot\vec{F}(\vec{x})\,dV \equiv \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)dx\,dy\,dz =$$

$$= \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz\,\Big(F_x(x_2,y,z) - F_x(x_1,y,z)\Big) + \ \dots \ = \int_S \vec{F}\cdot d\vec{S}$$

- Two useful equations of phenomenological fluid dynamics

+ continuity equation:

- for an arbitrary volume element, its density change is equal to the amount of mass flowing through its walls inside

$$\int_V \frac{\partial\rho}{\partial t}\,d^3x = - \int_S \rho\vec{v}\cdot d\vec{S}$$

normals of the surface point outwards

$$\int_V \frac{\partial\rho}{\partial t}\,d^3x + \int_V \nabla\cdot(\rho\vec{v})\,d^3x = 0$$ ← shrink the element into a single point

$$\rightarrow \underline{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

+ Euler's equation of motion

- E.M for an element of mass M:

$$M\frac{d\vec{v}}{dt} = -\int_S p\,d\vec{S} - M\nabla\Phi$$ ← $M=\int_V \rho\,d^3x$, divergence theorem

$$\int_V \rho\,d^3x\,\frac{d\vec{v}}{dt} = -\int_V \nabla p\,d^3x - \int_V \rho\,d^3x\,\nabla\Phi$$ ← shrink to a point

$$\rho\frac{d\vec{v}}{dt} = -\nabla p - \rho\nabla\Phi$$ ← use $\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \underbrace{\frac{\partial\vec{v}}{\partial x_j}\frac{dx_j}{dt}}$

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\,\vec{v}\cdot\nabla\vec{v} = -\nabla p - \rho\nabla\Phi$$

j - components: $\frac{\partial v_j}{\partial x_i}\frac{dx_i}{dt}$

- Distribution function

+ 6D phase space: $w_j = (\underbrace{x, y, z}_{q}, \underbrace{\dot{x}, \dot{y}, \dot{z}}_{p})$

+ distrib. f:

$$dN = f(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \underbrace{dx\,dy\,dz\,d\dot{x}\,d\dot{y}\,d\dot{z}}_{\text{phase space element}} dt$$

dN ↑ number of particles in the phase space element

+ conservation of fluid mass in the phase space

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial w_j}(f \dot{w}_j) = 0$$

+ Collisionless Boltzmann equation (CBE):

Hamilton eqs.

$$\frac{\partial}{\partial w_j}(f \dot{w}_j) = \underset{i=1\ldots3}{\frac{\partial}{\partial q_i}}(f \dot{q}_i) + \frac{\partial}{\partial p_i}(f \dot{p}_i) = \frac{\partial}{\partial q_i}\cdot\left(f \frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(f \frac{\partial H}{\partial q_i}\right) =$$

$$= \frac{\partial f}{\partial q_i} \underbrace{\frac{\partial H}{\partial p_i}}_{\dot{q}_i} + f \cancel{\frac{\partial^2 H}{\partial q_i \partial p_i}} - \frac{\partial f}{\partial p_i} \underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i} - f \cancel{\frac{\partial^2 H}{\partial p_i \partial q_i}} = \dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i}$$

$$\frac{\partial f}{\partial t} + \dot{w}_j \frac{\partial f}{w_j} = \frac{\partial f}{\partial t} + \dot{\vec{w}} \cdot \frac{\partial f}{\partial \vec{w}} = \frac{df}{dt} = 0$$

- collisionless Boltzmann equation (CBE)

  → collisionless system behaves as an incompressible fluid in the phase space

alternative form:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v_x} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial v_y} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0$$

• Jeans theorem

Integral of motion: $I(\vec{x},\vec{v})$: $\frac{d}{dt} I(\vec{x}(t), \vec{v}(t)) = 0$

using equations of motion:

$$\frac{dI}{dt} = \frac{\partial I}{\partial \vec{x}} \frac{d\vec{x}}{dt} + \frac{\partial I}{\partial \vec{v}} \frac{d\vec{v}}{dt} = \vec{v} \cdot \frac{\partial I}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial I}{\partial \vec{v}} = 0$$

$\Rightarrow$ condition to be an integral of motion is the same as the condition to be a steady-state solution of the CBE

(1) Any steady state solution of the CBE depends on the phase space coords only through the integrals of motion.

(2) Any function of the integrals of the integrals of motion is a steady state solution of the CBE

Proof:

ad 1: let f is s-s solution of CBE $\Rightarrow$ it is also an IoM. Q.E.D.

ad 2:
$$\frac{d}{dt} f\left(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), \dots I_n(\vec{x},\vec{v})\right) = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m} \underbrace{\frac{dI_m}{dt}}_{=0} = 0$$

• Jeans equations

- částicová hustota: $\nu \equiv \int f \, d^3v$ (mean particle density)
- 1 moment: $\overline{v_i} \equiv \frac{1}{\nu} \int f v_i \, d^3v$ (mean velocity)
- 2 moment: $\overline{v_i v_j} \equiv \frac{1}{\nu} \int f v_i v_j \, d^3v$
- n-th moment: $M^\alpha \equiv \frac{1}{\nu} \int v_i^{\alpha_1} v_l^{\alpha_2} v_s^{\alpha_3} \, d^3v$, $\alpha_1 + \alpha_2 + \alpha_3 = n$
- velocity dispersion: $\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v_i})(v_j - \overline{v_j})} = \overline{v_i v_j} - \overline{v_i}\,\overline{v_j}$

+ 0-th moment of CBE:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

$$\int \frac{\partial f}{\partial t}\, d^3v + \int \underbrace{v_i \frac{\partial f}{\partial x_i}}_{\text{indep.}}\, d^3v - \int \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\, d^3v = 0$$

$$\frac{\partial}{\partial t}\underbrace{\int f\, d^3v}_{\nu} + \frac{\partial}{\partial x_i}\underbrace{\int v_i f\, d^3v}_{\nu \bar{v}_i} - \frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\, d^3v = 0$$

- use divergence theorem: $\displaystyle\int_{V_v} \frac{\partial f}{\partial v_i}\, d^3v = \int_{S_v} f\, d^2S = 0$

assume v drops faster than $1/v^2$

$$\rightarrow \quad \underline{\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \bar{v}_i) = 0}$$

+ 1-st moment of CBE

$$\int v_j \frac{\partial f}{\partial t}\, d^3v + \int v_j v_i \frac{\partial f}{\partial x_i}\, d^3v - \int v_j \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v}$$

$$\frac{\partial}{\partial t}\int v_j \cancel{\partial} f\, d^3v + \frac{\partial}{\partial x_i}\int v_j v_i\, d^3v - \frac{\partial \Phi}{\partial x_i}\int \Big[\underbrace{\frac{\partial}{\partial v_i}(f v_j)}_{\substack{=0\\ \text{divergence}\\ \text{theorem}}} - f\frac{\partial v_j}{\partial v_i}\Big]\, d^3v = 0$$

$$\frac{\partial}{\partial t}(\nu \bar{v}_j) + \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j}) + \frac{\partial \Phi}{\partial x_i}\int f\, \delta_{ij}\, d^3v = 0$$

$$\underline{\frac{\partial}{\partial t}(\nu \bar{v}_j) + \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j}) + \nu\frac{\partial \Phi}{\partial x_j} = 0}$$

+ more common form of the 1st moment:

$$\frac{\partial}{\partial t}\left(\nu \bar{v}_j\right) + \frac{\partial}{\partial x_i}\left(\nu \overline{v_i v_j}\right) + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

$$\frac{\partial}{\partial x_i}\left(\nu \overline{v_i v_j}\right) = \frac{\partial}{\partial x_i}\left(\nu \sigma_{ij}^2\right) + \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i \bar{v}_j\right) = \frac{\partial}{\partial x_i}\left(\nu \sigma_{ij}^2\right) + \bar{v}_j \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right) + \nu \bar{v}_i \frac{\partial}{\partial x_i}\left(\bar{v}_j\right)$$

- subtract 0th moment $\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right) = 0 \;\Big|\; \times v_j$

$$\rightarrow \nu \frac{\partial v_j}{\partial t} + \frac{\partial}{\partial x_i}\left(\nu \sigma_{ij}^2\right) + \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \nu \frac{\partial \Phi}{\partial x_j} = 0$$