# Potential theory

Outline:

1. Derivation of the Poisson equation
2. Gauss theorem
3. Potential energy
4. Newton's shell theorems
5. Numerical calculation of grav. potential

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## 1. Poisson equation:

x ... position vector (no arrow);

F(x) ... force of gravity

$\Phi(x)$ ... grav. potential

gradient of inverted distance:

$$\nabla_x \left( \frac{1}{|x'-x|} \right) = \frac{x'-x}{|x'-x|^3}$$

Gravitational force (Coulomb's law):

$$F(x) = G M m \frac{x'-x}{|x'-x|^3}$$

Force on a unit mass (M=1) from the extended body with density distribution $\rho(x')$:

$$\delta F(x) = G \frac{x'-x}{|x'-x|^3} \delta m(x') = G \frac{x'-x}{|x'-x|^3} \rho(x') \delta^3 x'$$

$$F(x) = G \int \frac{x'-x}{|x'-x|^3} \rho(x') \, d^3x'$$

Let us define grav. potential: $\Phi(x) = -G \int \frac{\rho(x')}{|x'-x|} d^3x'$

Then $F(x) = \nabla_x \int \frac{G \rho(x')}{|x'-x|} d^3x' = -\nabla_x \Phi(x)$

- scalar field (grav. potential) can be defined only if the force field is conservative, i.e. the work requiered to get a mass from one position to another is independent of the path.
- scalar field is easier to visualize (not so much relevant today)
- scalar field is often easier to calculate (1/3 of the work)

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Let us take divergence of the force:

$$\nabla_x \cdot F(x) = G \int \nabla_x \cdot \left( \frac{x' - x}{|x' - x|^3} \right) \rho(x') \, d^3x'$$

$$\nabla_x \cdot \left( \frac{x' - x}{|x' - x|^3} \right) = - \frac{3}{|x' - x|^3} + \frac{3 (x' - x) \cdot (x' - x)}{|x' - x|^5}$$

$$= 0 \quad \text{for} \quad x \neq x'$$

$\Rightarrow$ integration can be restricted to a small volume $|x' - x| \leq h$

$$\nabla_x \cdot F(x) = G \rho(x) \int_{|x'-x| \leq h} \nabla_x \cdot \left( \frac{x' - x}{|x' - x|^3} \right) d^3x' =$$

symmetry

$$= -G \rho(x) \int_{|x'-x| \leq h} \nabla_{x'} \cdot \left( \frac{x' - x}{|x' - x|^3} \right) d^3x' =$$

divergence theorem

$$= -G \rho(x) \int_{|x'-x| = h} \frac{(x' - x) \cdot d^2 S'}{|x' - x|^3} =$$

integration on the surface of the sphere $|x' - x| = h$, $d^2S = (x' - x) h \, d^2\Omega$

$$\nabla_x \cdot F(x) = -G \rho(x) \int d^2\Omega = -4\pi G \rho$$

substituting $F = -\nabla\phi$ into $\nabla \cdot F$ :

$$\boxed{\nabla^2 \Phi = 4\pi G \rho}$$ - Poisson equation

special case in vacuum: $\nabla^2 \Phi = 0$ (Laplace equation)

- Grav. potential can be found by solving the Poisson equation (2nd order PDE); boundary conditions have to be known (often called Boundary value problem)

- Opposite direction: Poisson eq. can be solved by applying Green's function method and converting it to integral:

$L G(x', x) = \delta(x' - x)$ - definition of Green's function of operator $L$

in our case: $L \equiv \nabla^2 \Rightarrow \nabla^2 \frac{1}{|x' - x|} = -4\pi\, \delta(x' - x)$

$$\left.\begin{array}{l} L u = f \\ D u = 0 \text{ (boundaries)} \end{array}\right\} \Rightarrow u(x) = \int f(x')\, G(x, x')\, d^3x'$$

in our case: $\Phi(x) = -G \int \frac{\rho(x')}{|x' - x|}\, d^3x'$

## 2. Gauss theorem

- integrating Poisson eq. over an arbitrary volume:

$$\int_V \nabla^2 \Phi \, d^3x = 4\pi G \int_V \rho \, d^3x$$

$$\int_{\partial S} \nabla \Phi \, d^2S = 4\pi G M$$

The integral of the normal component of $\nabla\Phi$ over any closed surface equals $4\pi G$ times the mass contained within that surface.

## 3. Potential energy

- work needed to assemble the system (by bringing its mass from infinity)

assume: • some mass already in place: $\rho(x)$, $\Phi(x)$
• bringing mass element $\delta m$ => work $\delta m \, \Phi(x)$

$$\delta W = \int \delta\rho(x) \, \Phi(x) \, d^3x$$

using PE: $\delta\Phi = 4\pi G \, \delta\rho$

$$\delta W = \frac{1}{4\pi G} \int \Phi(x) \, \nabla^2(\delta\Phi) \, d^3x$$

applying per partes: $\int_V g \, \nabla \cdot \vec{F} \, d^3x = \int_S g \, \vec{F} \cdot d\vec{S} - \int_V (F \cdot \nabla) g \, d^3x$

$$\delta W = \underbrace{\frac{1}{4\pi G} \int \Phi \, \nabla(\delta\Phi) \cdot dS}_{\to 0 \text{ for } r \to \infty} - \frac{1}{4\pi G} \int \underbrace{\nabla\Phi \cdot \nabla(\delta\Phi)}_{= \frac{1}{2}\delta|\nabla\Phi|^2} \, d^3x$$

$$\delta W = -\frac{1}{8\pi G}\,\delta \int |\nabla \Phi|^2 d^3x$$

$$W = -\frac{1}{8\pi G}\int |\nabla\Phi|^2 d^3x = +\frac{1}{8\pi G}\int \Phi\,\nabla^2\Phi\, d^3x - \underbrace{\frac{1}{8\pi G}\int \Phi\nabla\Phi\cdot d^2S}_{0}$$

$$\boxed{W = \frac{1}{2}\int \rho(x)\,\Phi(x)\,d^3x}$$

## Ghandrasekhar potential energy tensor

$$W_{jk} = -\int \rho(x)\,x_j\,\frac{\partial \Phi}{\partial x_k}\,d^3x$$

inserting $\Phi(x) = -G\int \rho(x')\frac{1}{|x'-x|}\,d^3x'$

$$W_{jk} = G\int_V \rho(x)\,x_j\,\frac{\partial}{\partial x_k}\int_{V'}\frac{\rho(x')}{|x'-x|}\,d^3x'\,d^3x$$

$V'$ ← does not depend on x

$$W_{jk} = G\int_V\int_{V'} \rho(x)\,\rho(x')\,\frac{x_j(x_k'-x_k)}{|x'-x|^3}\,d^3x'\,d^3x \qquad (1)$$

x and x' are dummy variables of integration, can be relabeled: $x \leftrightarrow x'$ :

$$W_{jk} = \int_{V'}\int_V \rho(x')\,\rho(x)\,\frac{x_j'(x_k-x_k')}{|x-x'|^3}\,d^3x\,d^3x' \qquad (2)$$

$(1) + (2) = 2\,W_{jk}$

$$x_j(x_k'-x_k) + x_j(x_k - x_k') = (x_j - x_j')(x_k' - x_k)$$

$$W_{jk} = -\frac{1}{2}\int\int \rho(x)\,\rho(x')\,\frac{(x_j'-x_j)(x_k'-x_k)}{|x-x'|^3}\,d^3x'\,d^3x \quad \text{- \underline{symmetric}}$$

(6)

$$\mathrm{trace}(\mathcal{W}) = \sum_{j=1}^{3} \mathcal{W}_{jj} = -\frac{1}{2} G \int_V \rho(x) \underbrace{\int_{V'} \frac{\rho(x')}{|x'-x|} d^3x'}_{-\Phi(x)/G} d^3x$$

$$\mathrm{trace}(\mathcal{W}) = W \equiv \frac{1}{2} \int \rho(x)\, \Phi(x)\, d^3x$$

- Spherical distribution :

$$W = -\int \rho\, x \cdot \nabla\Phi\, d^3x \qquad \text{(trace of definition of } \mathcal{W}_{jk}\text{)}$$

$$\nabla\Phi = \frac{d\Phi}{dr} \vec{e}_r = \frac{G M(r)}{r^2} \vec{e}_r$$

$$W = -4\pi G \int_0^\infty \rho\, M(r)\, r\, dr \quad ; \quad \mathcal{W}_{jk} = \frac{1}{3} W \delta_{jk}$$

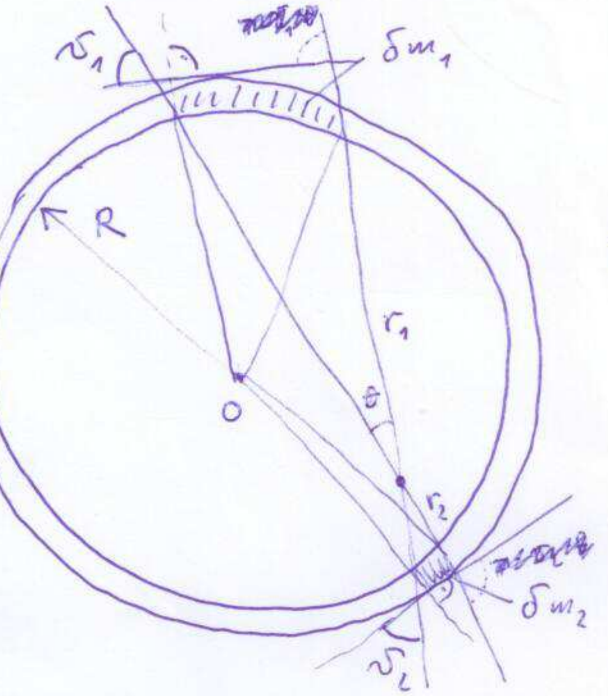
- isotropic (from symmetry)

- Gravitational radius
  - characterizes system without a sharp boundary (e.g. stellar cluster)

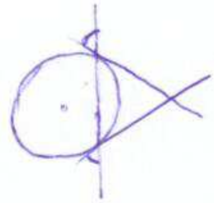
$$r_g \equiv \frac{G M^2}{|W|}$$

# 4. Newton's shell theorems

First: A body that is inside a spherical shell of matter experiences no net gravitational force from the shell




$$\vartheta_1 = \vartheta_2$$



$$\Downarrow$$

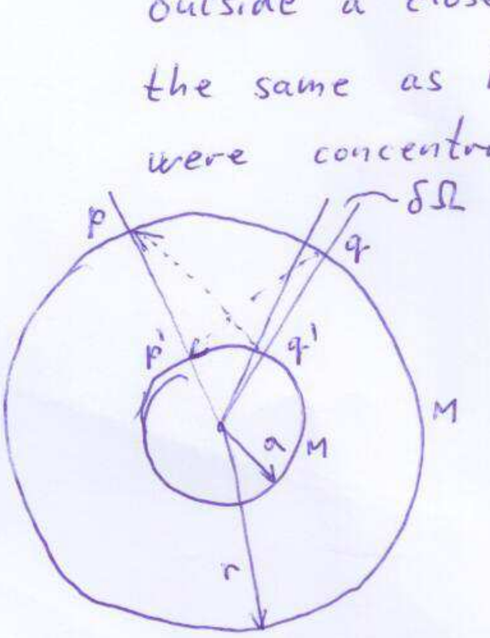
$$\frac{(r_1\theta)^2}{\cos\vartheta_1} \approx$$

$$\frac{\delta m_1}{\delta m_2} = \left(\frac{r_1}{r_2}\right)^2$$

$$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$

$\Rightarrow$ $\Phi$ constant inside the shell; can be estimated easily in the centre: $\Phi = -G\int\frac{\delta m}{r} = -\frac{GM}{R}$

Second: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its centre




$$\delta\Phi = -\frac{GM}{|p-q'|}\frac{\delta\Omega}{4\pi}$$

- contribution to the potential at $p$ from the shell element at $q'$

$$\delta\Phi' = -\frac{GM}{|p'-q|}\frac{\delta\Omega}{4\pi}$$

- contribution to the potential at $p'$ from the shell element at $q$

since $|p-q'| = |p'-q| \Rightarrow \delta\Phi = \delta\Phi'$

by integrating over all $\delta\Omega$ elements: $\Phi = \Phi'$

From the first theorem, we know potential inside the larger shell $\Phi' = -\frac{GM}{r}$; therefore $\Phi = -\frac{GM}{r}$

Q.E.D.

- took Newton more than 10 years

- potential of an arbitrary spherically symmetric density distribution:

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\underbrace{\int_0^r \rho(r')r'^2\,dr'}_{M(r)} + \int_r^\infty \rho(r')r'\,dr'\right]$$

- two important properties of spherically symmetric density distribution:

$$v_c^2 = r\frac{d\Phi}{dr} = r|F| = \frac{GM(r)}{r} \text{ - circular speed}$$

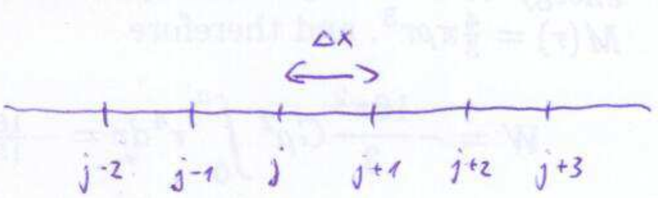
$$v_e(r) = \sqrt{2|\Phi(r)|} \text{ - escape speed}$$

# 5. Numerical calculation of grav. potential

(9)

Let us discretize Poisson equation

$\Phi_j \;,\; x_j \;,\; \rho_j$




$$\frac{d^2\Phi}{dx^2} = 4\pi G \rho$$

$$\left.\frac{d\Phi}{dx}\right|_{j+1/2} = \frac{\Phi_{j+1} - \Phi_j}{\Delta x} \quad , \quad \left.\frac{d^2\Phi}{dx^2}\right|_j = \frac{\left.\frac{d\Phi}{dx}\right|_{j+1/2} - \left.\frac{d\Phi}{dx}\right|_{j-1/2}}{\Delta x} =$$

$$= \frac{\frac{\Phi_{j+1} - \Phi_j}{\Delta x} - \frac{\Phi_j - \Phi_{j-1}}{\Delta x}}{\Delta x} =$$

$$= \frac{1}{(\Delta x)^2}\left( \Phi_{j+1} - 2\Phi_j + \Phi_{j-1} \right)$$

$$\underline{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1} = 4\pi G (\Delta x)^2 \rho_j}$$

- 1D : Gauss elimination of tri-diagonal matrix
- iterative methods

$$\Phi_j^{(1)} = \frac{1}{2}\left( \Phi_{j-1}^{(0)} + \Phi_{j+1}^{(0)} + 4\pi G (\Delta x)^2 \rho_j \right) < \begin{matrix} \text{Jacobi} \\ \text{Gauss-Seidel} \end{matrix}$$

  - slow for long modes of error → multigrid
- tree codes
  - integration over all volume, matter further away grouped into tree nodes (Barnes & Hut 1986), (Salmon & Warren 1994)
- spectral methods : FFT, spherical harmonics
  - needs uniform grid of size $2^n$
- direct $N^2$ integration : N body simulations, needs special hardware GPUs