1. Modified Boyle's law by Terletsky

$$pV = NkT$$ - Boyle's law

$$pV = NkT - \alpha G M^2 V^{-1/3}$$ - correction for grav. energy (Terletsky, 1952)

volume fluctuations of the ideal gas given by:

$$\frac{\overline{(V-\bar{V})^2}}{\bar{V}^2} = -\frac{kT}{V^2\left(\frac{\partial p}{\partial V}\right)_T}$$

(Statistical Physics Landau & Lifshitz §112 Fluctuations of the fundamental thermodyn. q. eq. 112.7)

$$p = \frac{NkT}{V} - \alpha\frac{GM^2}{V^{4/3}}$$

$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{NkT}{V^2} + \frac{4\alpha}{3}\frac{GM^2}{V^{7/3}} \quad \leftarrow M = mN$$

$$\frac{\overline{(V-\bar{V})^2}}{\bar{V}^2} = \frac{1}{N}\left[1 - \frac{4}{3}\frac{\alpha G m^2 N}{kT V^{1/3}}\right]^{-1}$$

fluktuations large if
$$\left[\begin{array}{l} N \text{ is small} \\ \frac{M}{R} \equiv \frac{mN}{V^{1/3}} \approx \frac{3kT}{4\alpha G m} \end{array}\right.$$

2. Hydrostatic equilibrium eq.

$$-\frac{dp}{dr} = \frac{4\pi G \rho}{r^2}\int_0^r \rho y^2 dy$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho \qquad + \qquad p = \frac{\rho k T}{m}$$

→ Emden's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{4\pi G m \rho}{kT}$$

→ in a form of the Boyle's law:

$$N = \frac{4\pi}{m}\int_0^r \rho y^2 dy$$

$$p = \frac{\rho k T}{m}$$

$$pV - NkT = \frac{4\pi}{3} r^3 p - \frac{4\pi k T}{m}\int_0^r \rho y^2 dy =$$

$$= \frac{4\pi}{3}\int_0^r y^3 \frac{dp}{dy} dy$$

$$pV = NkT - \frac{16}{3}\pi^2 G \int_0^r \rho y dy \int_0^y \rho x^2 dx$$

for $\rho = \text{const}$ (which does not satisfy hydrst. equil.) we get:

$$\alpha = \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3}$$

3. Transformation of Emden's equation:

$$r, \rho, T, m \longrightarrow \xi, \psi, \lambda, \beta$$

$$r = \beta^{1/2} \lambda^{-1/2} \xi \, , \quad \rho = \lambda \exp(-\psi) \, , \quad \beta = \frac{kT}{4\pi G m}$$

Boundary cond. at $r = 0$:

$$\left.\begin{aligned} \rho &= \rho_0 \\ \frac{d\rho}{dr} &= 0 \end{aligned}\right. \quad \Longrightarrow \quad \begin{aligned} \lambda &= \rho_c \\ \psi &= 0 \\ \frac{d\psi}{d\xi} &= 0 \end{aligned}$$

Transformed equation:

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{d\rho}{dr} \right) = - \frac{4\pi G m \rho}{kT}$$

$$\cancel{\beta^{-1}} \cancel{\lambda} \xi^{-2} \cancel{\beta^{-1/2}} \cancel{\lambda^{1/2}} \frac{d}{d\xi} \left(\cancel{\beta} \cancel{\lambda^{-1}} \xi^2 \cancel{\lambda^{-1}} \exp(+\psi) \cancel{\beta^{-1/2}} \cancel{\lambda^{1/2}} \frac{d}{d\xi} \left(\cancel{\lambda} \exp(-\psi) \right) \right) =$$

$$= - \frac{4\pi G m}{kT} \overset{\cancel{\beta^{-1}}}{} \cancel{\lambda} \exp(-\psi)$$

$$\xi^{-2} \frac{d}{d\xi} \left[\cancel{\exp(+\psi)} \cancel{\exp(-\psi)} \left(+ \frac{d\psi}{d\xi} \right) \right] = + \exp(-\psi)$$

$$\xi^{-2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi}{d\xi} \right) = \exp(-\psi)$$

$\lambda \equiv \rho_c$ ..... parameter of the family of isothermal spheres

4. Construction of the p-V curve

- spherical container with volume V_0
- fixed number of particles N_0
- pressure at the boundary p_0
- constant temperature T

- varying $p_0 \rightarrow$ variation of V_0 and λ
- impossible to find formula $p_0(V_0, N_0, T)$
- however, $\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0, T}$ can be found

$$N = \frac{4\pi}{m} \int_0^r \rho y^2 dy \leftarrow , \quad -\frac{dp}{dr} = \frac{4\pi G \rho}{r^2} \int_0^r \rho y^2 dy$$

$$N = \frac{4\pi}{m}\left(-\frac{dp}{dr}\right)\frac{r^2}{4\pi G \rho} = -\frac{4\pi}{m}\frac{kT}{m}\frac{r^2}{4\pi G \rho}\frac{d\rho}{dr}$$

$$= -\frac{kT}{Gm^2}\beta \lambda^{-1} \xi \beta^{2-1/2} \lambda^{1/2} \frac{d}{d\xi}\left(\lambda \exp(-\psi)\right) \lambda^{-1} \exp(\psi) \Rightarrow$$

$$\boxed{N = \frac{4\pi}{m} \beta^{3/2} \lambda^{-1/2} \xi^2 \frac{d\psi}{d\xi}} \qquad (2.9) \leftarrow \text{hydrost. equil. included in this eq.}$$

$$\delta N_0 = \frac{4\pi}{m} \beta^{3/2} \underbrace{\delta\left\{\lambda^{-1/2} \xi_0^2 \left(\frac{d\psi}{d\xi}\right)_0\right\}}_{=\emptyset} = \emptyset$$

- const. numb. of particles
- suffix 0 means "at the boundary"

$$\Rightarrow -\frac{1}{2}\lambda^{-3/2} \xi_0^2 \left(\frac{d\psi}{d\xi}\right)_0 \delta\lambda + \lambda^{-1/2} \underbrace{\left\{\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right)\right\}_0}_{=\xi_0^2 \exp(-\psi_0)} \delta \xi_0 = \emptyset$$

Transform: $\lambda, \xi_0 \rightarrow \lambda, r_0$

$$\delta \xi_0 = \left(\frac{\partial \xi_0}{\partial r_0}\right)_\lambda \delta r_0 + \left(\frac{\partial \xi_0}{\partial \lambda}\right)_{r_0} \delta \lambda$$

$$\xi_0 = \beta^{-1/2}\lambda^{1/2} r_0 \qquad (2.6)$$

⑤

$$\Rightarrow \quad \frac{\partial \xi_0}{\partial r_0} = \beta^{-1/2}\lambda^{1/2} \quad , \quad \frac{\partial \xi_0}{\partial \lambda} = \frac{1}{2}\beta^{-1/2}\lambda^{-1/2} r_0 = \frac{1}{2}\lambda^{-1}\xi_0$$

$$-\frac{1}{2}\lambda^{-3/2}\xi_0^2\left(\frac{d\psi}{d\xi}\right)_0 \delta\lambda + \lambda^{-1/2}\xi_0^2 \exp(-\psi_0)\,\delta\xi_0 = \emptyset$$

$$\delta\lambda\left[-\frac{1}{2}\lambda^{-3/2}\xi_0^2\left(\frac{d\psi}{d\xi}\right)_2 + \frac{1}{2}\lambda^{-3/2}\xi_0^3\exp(-\psi_0)\right] + \beta^{-1/2}\xi_0^2\exp(-\psi_0)\,\delta r_0 = \emptyset$$

$$\delta\lambda = \frac{-\beta^{-1/2}\xi_0^2\exp(-\psi_0)\,\delta r_0}{-\frac{1}{2}\lambda^{-3/2}\xi_0^2\left(\frac{d\psi}{d\xi}\right)_0 + \frac{1}{2}\lambda^{-3/2}\xi_0^3\exp(-\psi_0)}$$

$$\xi_0, \psi_0, \beta \rightarrow r_0, \rho_0, N_0$$

$$\xi_0 = \beta^{-1/2}\lambda^{1/2} r_0$$

$$\exp(-\psi_0) = \rho_0 \lambda^{-1}$$

$$N_0 = \frac{4\pi}{m}\beta^{3/2}\lambda^{-1/2}\xi_0^2\left(\frac{d\psi}{d\xi}\right)_0 \qquad (2.9)$$

$$\delta\lambda = \frac{-\cancel{\beta^{-1/2}}\,\cancel{\beta^{-1}}\,\lambda\, r_0^2\,\rho_0\,\cancel{\lambda^{-1}}\,\delta r_0}{-\frac{1}{2}\cancel{\lambda^{-3/2}}\,N_0\frac{m}{4\pi}\cancel{\beta^{-3/2}}\,\cancel{\lambda^{1/2}} + \frac{1}{2}\cancel{\lambda^{-3/2}}\,\cancel{\beta^{-3/2}}\,\cancel{\lambda^{3/2}}\,r_0^3\,\rho_0\,\cancel{\lambda^{-1}}}$$

$$= \frac{-2r_0^2\lambda\,\rho_0\,\delta r_0}{-\frac{mN_0}{4\pi} + r_0^3\rho_0}$$

$$\delta\lambda = \frac{2\lambda\rho_0\,\delta r_0}{\frac{mN_0}{4\pi r_0^2} \mp r_0\rho_0}$$

$$\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = \frac{kT}{m}\left(\frac{\partial \rho_0}{\partial V_0}\right)_{N_0,T} = \frac{kT}{4\pi m r_0^2}\left(\frac{\partial \rho_0}{\partial r_0}\right)_{N_0,T}$$

$$V = \frac{4\pi}{3} r^3 \rightarrow \frac{\partial V}{\partial r} = \frac{4\pi}{\cancel{3}} \cancel{3} r^2$$

$$\delta\rho_0 = \delta\left(\lambda \exp(-\psi_0)\right) = \exp(-\psi_0)\,\delta\lambda - \lambda\exp(-\psi_0)\left(\frac{d\psi}{d\xi}\right)_0 \delta\xi_0 =$$

$$\cancel{\delta\xi_0 = \beta^{-1/2}\lambda^{1/2}\delta r_0}$$

$$= \exp(-\psi_0)\left\{\delta\lambda\left[1 - \lambda\left(\frac{d\psi}{d\xi}\right)_0 \underbrace{\frac{\partial\xi_0}{\partial\lambda}}_{\frac{1}{2}\lambda^{-1}\xi_0}\right] - \lambda\left(\frac{d\psi}{d\xi}\right)_0 \underbrace{\frac{\partial\xi_0}{\partial r_0}}_{\beta^{-1/2}\lambda^{1/2}}\delta r_0\right\}$$

$$\left(\frac{d\psi}{d\xi}\right)_0 = \frac{mN_0}{4\pi}\beta^{-3/2}\lambda^{1/2}\xi_0^{-2} = \frac{mN_0}{4\pi r_0^2}\beta^{-1/2}\lambda^{-1/2}$$

$$\delta\rho_0 = \exp(-\psi_0)\left\{\delta\lambda\left[1 - \frac{mN_0}{8\pi r_0^2}\beta^{-1/2}\lambda^{-1/2}\underbrace{\xi_0}_{\beta^{-1/2}\lambda^{1/2}r_0}\right] - \lambda\frac{mN_0}{4\pi r_0^2}\beta^{-1/2}\lambda^{-1/2}\beta^{-1/2}\lambda^{1/2}\delta r_0\right\}$$

$$\delta\rho_0 = \exp(-\psi_0)\left\{\delta\lambda\left[1 - \frac{mN_0}{8\pi\beta r_0}\right] - \frac{mN_0\lambda}{4\pi r_0^2\beta}\delta r_0\right\}$$

$$\rho_0\lambda^{-1} \qquad\qquad \delta\lambda = \frac{2\lambda\rho_0}{\frac{mN_0}{4\pi r_0^2} - r_0\rho_0}\delta r_0$$

$$\delta\rho_0 = \rho_0\lambda^{-1}\left\{\frac{2\lambda\rho_0}{\frac{mN_0}{4\pi r_0^2} - r_0\rho_0}\left[1 - \frac{mN_0}{8\pi\beta r_0}\right] - \frac{mN_0\lambda}{4\pi r_0^2\beta}\right\}\delta r_0$$

$$p_0 = \frac{\rho_0 kT}{m}, \quad \beta = \frac{kT}{4\pi G m}, \quad r_0 = \left[\frac{3}{4\pi}V_0\right]^{1/3}$$

$$\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = \frac{kT}{4\pi m r_0^2}\,\frac{\rho_0}{r_0}\,\frac{r_0}{\rho_0}\left(\frac{\partial \rho_0}{\partial r_0}\right)_{N_0,T} = \frac{p_0}{3V_0}\,\frac{r_0}{\rho_0}\left(\frac{\partial \rho_0}{\partial r_0}\right)_{N_0,T} =$$

$$= \frac{2p_0}{3V_0}\left\{\frac{\rho_0 r_0}{\frac{mN_0}{4\pi r_0^2} - r_0\rho_0}\left[1 - \frac{mN_0}{8\pi \underset{\overset{\|}{\frac{kT}{4\pi Gm}}}{\beta} r_0}\right] - \frac{mN_0}{8\pi r_0^3 \beta}\right\} =$$

$$= \frac{2p_0}{3V_0}\left\{\frac{r_0\rho_0\left[1 - \frac{Gm^2N_0}{2kTr_0}\right]}{\frac{mN_0 - 4\pi r_0^3\rho_0}{4\pi r_0^2}} - \frac{mN_0\cdot 4\pi Gm}{2\,\cancel{8\pi} r_0^4 kT}\right\} =$$

$$= \frac{2p_0}{3V_0}\left\{\frac{4\pi r_0^3\rho_0\left(2kTr_0 - Gm^2N_0\right)}{2kTr_0\left(mN_0 - 4\pi r_0^3\rho_0\right)} - \frac{Gm^2N_0}{2r_0^4 kT}\right\} =$$

$$= \frac{2p_0}{3V_0}\,\frac{8\pi r_0^4\rho_0 kT - \cancel{4\pi r_0^3\rho_0 Gm^2N_0} - Gm^3N_0^2 + \cancel{4\pi r_0^3\rho_0 Gm^2N_0}}{2kTr_0\left(mN_0 - 4\pi r_0^3\rho_0\right)} =$$

$$= -\frac{2p_0}{3V_0}\,\frac{Gm^3N_0^2 - 8\pi r_0^4\rho_0 kT}{2kTr_0 mN_0 - 8\pi r_0^4\rho_0 kT} = \qquad 4\pi r_0^4 = 3V_0\left(\frac{3V_0}{4\pi}\right)^{1/3}$$

$$= -\frac{2p_0}{3V_0}\,\frac{1 - \frac{Gm^3N_0^2}{8\pi r_0^4 \underbrace{\rho_0 kT}_{p_0}}}{1 - \frac{2kTr_0 mN_0}{4\cdot 2\pi r_0^4 \underbrace{\rho_0 kT}_{p_0}}} = -\frac{2p_0}{3V_0}\,\frac{1 - \frac{Gm^2N_0^2}{8\pi r_0^4 p_0}}{1 - \frac{N_0kT}{\underbrace{4\pi r_0^3}_{3V_0} p_0}} \Rightarrow$$

$$\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = -\frac{2p_0}{3V_0}\,\frac{1 - \left(\frac{4\pi}{3}\right)^{1/3}\frac{Gm^2N_0^2}{6p_0V_0^{4/3}}}{1 - \frac{N_0kT}{3p_0V_0}}$$

for $G = 0$: $\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = -\frac{2p_0}{3V_0}\,\frac{1}{\frac{3p_0V_0 - NkT}{3p_0V_0}} = -\frac{2N_0kT}{3V_0^2}\,\frac{3N_0kT}{2N_0kT} = -\frac{N_0kT}{V_0^2}$

5. Obtaining p and V from Emden's eq. solution

- p and V as functions of ξ, Ψ and $\frac{d\Psi}{d\xi}$

$$N = \frac{4\pi}{m} \beta^{3/2} \lambda^{-1/2} \xi^2 \frac{d\Psi}{d\xi}$$

$$\lambda^{1/2} = \frac{4\pi}{mN} \beta^{3/2} \xi^2 \frac{d\Psi}{d\xi}$$

$$p = \frac{\rho k T}{m} = \lambda \exp(-\Psi) \frac{kT}{m} = \frac{16\pi^2}{m^2 N^2} \left(\frac{kT}{4\pi G m}\right)^3 \xi^4 \left(\frac{d\Psi}{d\xi}\right)^2 \exp(-\Psi) \frac{kT}{m}$$

$$p = \frac{k^4 T^4}{4\pi G^3 m^6 N^2} \xi^4 \left(\frac{d\Psi}{d\xi}\right)^2 \exp(-\Psi)$$

$$V = \frac{4\pi}{3} r^3 = \frac{4\pi}{3} \beta^{3/2} \lambda^{-3/2} \xi^3 = \frac{4\pi}{3} \beta^{3/2} \left(\frac{mN}{4\pi}\right)^3 \beta^{-9/2} \xi^{-6} \left(\frac{d\Psi}{d\xi}\right)^{-3} \xi^3$$

$$= \frac{4\pi}{3} \left(\frac{4\pi G m}{kT}\right)^3 \left(\frac{mN}{4\pi}\right)^3 \xi^{-3} \left(\frac{d\Psi}{d\xi}\right)^{-3}$$

$$V = \frac{4\pi}{3} \left(\frac{G m^2 N}{kT \xi \frac{d\Psi}{d\xi}}\right)^3$$

6. Singular point of eq. for $\frac{\partial p_0}{\partial V_0}$

$$1 - \left(\frac{4\pi}{3}\right)^{1/3} \frac{G m^2 N_0^2}{6 p_0 V_0^{4/3}} = 0$$

$$1 - \frac{N_0 k T}{3 p_0 V_0} = 0 \longrightarrow p_0 = \frac{N_0 k T}{3 V_0}$$

$$1 = \left(\frac{4\pi}{3}\right)^{1/3} \frac{G m^2 N_0^{\cancel{2}} \cdot 3 V_0}{{}_2\cancel{6} N_0 k T V_0^{4/3}}$$

$$V_0 = \frac{\pi}{6} \left(\frac{G m^2 N_0}{kT}\right)^3$$

$$p_0 = \frac{2}{\pi} \frac{k^4 T^4}{G^3 m^6 N_0^2}$$

7. Stability

$$1 - \frac{\left(\frac{4\pi}{3}\right)^{1/3} G m^2 N^2}{6 p V^{4/3}} = 1 - \frac{\cancel{\left(\frac{4\pi}{3}\right)^{1/3}} G m^2 N^2}{6 \frac{\cancel{k^4 T^4}}{4\pi G^3 m^6 N^2} \xi^4 \left(\frac{d\psi}{d\xi}\right)^2 \exp(-\psi) \left(\frac{4\pi}{3}\right)^{4/3} \left(\frac{G m^2 N}{\cancel{kT} \cancel{\xi \frac{d\psi}{d\xi}}}\right)}$$

$$= 1 - \frac{G m^2 N^2 \left(\frac{d\psi}{d\xi}\right)^2 \exp(\psi)}{2 G m^2 N^2} = 1 - \frac{1}{2} \exp(\psi) \left(\frac{d\psi}{d\xi}\right)^2 \qquad (3.1)$$

$$1 - \frac{kTN}{3pV} = 1 - \frac{\cancel{kTN}}{\cancel{3} \frac{\cancel{k^4 T^4}}{4\pi G^3 m^6 N^2} \xi^4 \left(\frac{d\psi}{d\xi}\right)^2 \exp(-\psi) \cancel{\left(\frac{4\pi}{3}\right)} \left(\frac{\cancel{G m^2 N}}{\cancel{kT} \xi \left(\frac{d\psi}{d\xi}\right)}\right)^3} =$$

$$= 1 - \xi^{-1} \exp(\psi) \frac{d\psi}{d\xi} \qquad (3.2)$$

$$\left(\frac{\partial p}{\partial V}\right)_{N,T} \equiv \frac{(3.1)}{(3.2)}$$

both (3.1) and (3.2) are positive for $\xi < 6{,}5$

critical radius : $r_{crit} = 6{,}5\, \beta^{1/2} \lambda^{-1/2} = 6{,}5 \left(\frac{kT}{4\pi G m \lambda}\right)^{1/2}$

$$r_{crit} = 1{,}8 \left(\frac{kT}{m G \rho_{cen}}\right)^{1/2}$$

integrating ρ :

$$\frac{M}{r_{crit}} = 2{,}4 \frac{kT}{m G}$$

compare to Jeans relation :

$$\frac{M}{r_{Jeans}} = 3{,}3 \frac{kT}{m G}$$

Summary:

Emden eq.: $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{4\pi G m \rho}{kT}$

Transformation: $r, \rho, T, m \rightarrow \xi, \psi, \lambda, \beta$

$$r = \beta^{1/2}\lambda^{-1/2}\xi \,,\quad \rho = \lambda \exp(-\psi) \,,\quad \beta = \frac{kT}{4\pi G m}$$

$$\rightarrow \xi^{-2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = \exp(-\psi)$$

+ boundary cond at $r = \emptyset$: $\lambda = \rho_c \,,\ \frac{d\psi}{d\xi} = 0 \,,\ \psi = \emptyset$

Variations at the boundary for fixed N_0:

$$N_0 = \frac{4\pi}{m}\beta^{3/2}\lambda^{-1/2}\xi_0^2\left(\frac{d\psi}{d\xi}\right)_0$$ – From hydrost. equil.

$\rightarrow \lambda^{1/2} = \frac{4\pi}{mN}\beta^{3/2}\xi^2\frac{d\psi}{d\xi}$

– to determine $\lambda \equiv \rho_c$

– ξ and ψ obtained from P-V diagram

$$\delta N_0\left(\lambda, \xi_0, \left(\frac{d\psi}{d\xi}\right)_0\right) = \emptyset$$

$$\underline{\;\cdots\;}\,\delta\lambda + \underline{\;\cdots\;}\,\delta\xi_0 = \emptyset \xrightarrow[\xi_0 \to r_0]{} \underline{\;\cdots\;}\,\delta\lambda + \underline{\;\cdots\;}\,\delta r_0 = \emptyset$$

$$\rightarrow \delta\lambda = \underline{\;\cdots\;}\,\delta r_0$$

$\xi_0, \psi_0, \beta \rightarrow r_0, \rho_0, N_0$

P-V curve: $\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = \frac{kT}{m}\left(\frac{\partial \rho_0}{\partial V_0}\right)_{N_0,T} = \frac{kT}{4\pi m r_0^2}\left(\frac{\partial \rho_0}{\partial r_0}\right)_{N_0,T}$

$$\delta\rho_0 = \delta\left(\lambda \exp(-\psi_0)\right) = \underline{\;\cdots\;}\,\delta\lambda + \underline{\;\cdots\;}\,\delta\xi_0 \xrightarrow[\xi_0 \to r_0]{}$$

$$= \underline{\;\cdots\;}\,\delta\lambda + \underline{\;\cdots\;}\,\delta r_0 \rightarrow \delta\rho_0 = \underline{\;\cdots\;}\,\delta r_0$$

$$\left(\frac{\partial p_0}{\partial V_0}\right)_{N_0,T} = -\frac{2p_0}{3V_0}\,\frac{1 - \left(\frac{4\pi}{3}\right)^{1/3}\frac{G m^2 N_0^2}{6 p_0 V_0}}{1 - \frac{N_0 k T}{3 p_0 V_0}}$$