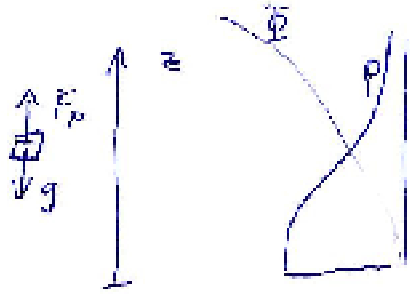
1. Isothermal atmosphere

$$-\nabla\Phi = -g = \text{const}, \quad P = c_s^2\rho$$



$$\frac{\partial P}{\partial z} = -\rho\frac{\partial \Phi}{\partial z} = -\rho g$$

$$c_s^2\frac{\partial \rho}{\partial z} = -\rho g$$

$$\int\frac{d\rho}{\rho} = -\frac{g}{c_s^2}\int dz$$

$$\ln\rho = -\frac{g}{c_s^2}(z - z_0)$$

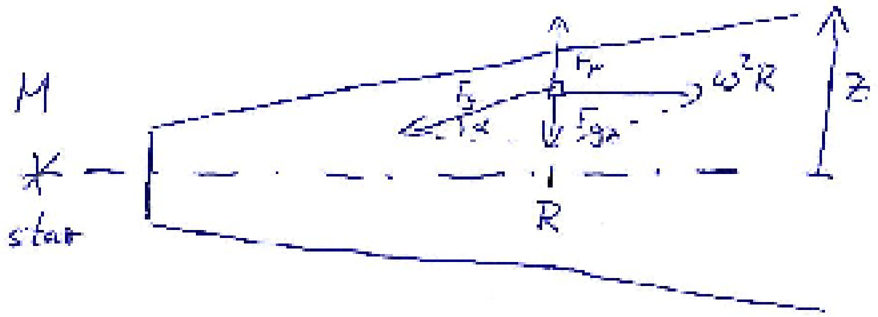
$$\rho = \rho_0 \exp\left(-\frac{gz}{c_s^2}\right)$$

$$\vec{F}_p + \vec{F}_g = 0$$

$$-\nabla P - \rho\nabla\Phi = 0$$

$$-\nabla P - \rho g = 0$$

2. Vertical structure of isothermal accretion disc



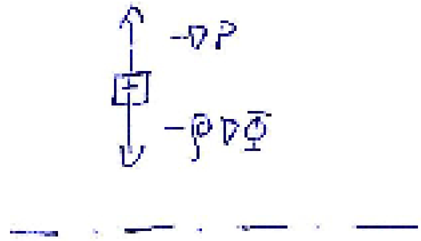
$$\frac{\partial P}{\partial z} = -F_{gz} = -F_g\, \mathrm{tg}\,\alpha = -\frac{GM\rho}{R^2}\frac{z}{R}$$

$$c_s^2\frac{\partial \rho}{\partial z} = -\frac{GM}{R^3}\rho z$$

$$\ln\rho = -\frac{GM}{2c_s^2R^3}(z^2 - z_0^2)$$

$$\rho = \rho_0 \exp\left(-\frac{GM}{2c_s^2R^3}z^2\right)$$

3. Isothermal self-gravitating slab



$$\frac{\partial P}{\partial z} = -\rho\,\frac{\partial \Phi}{\partial z}$$

$$\frac{c_s^2}{\rho}\,\frac{\partial \rho}{\partial z} = -\,\frac{\partial \Phi}{\partial z}$$

$$c_s^2\,\frac{\partial}{\partial z}(\ln \rho) = -\,\frac{\partial \Phi}{\partial z}$$

$$\Rightarrow \Phi = -c_s^2 \ln \rho + \text{const} = -c_s^2 \ln\frac{\rho}{\rho_0} + \Phi_0$$

$$\rho_0 = \rho(z=0)\ ,\quad \Phi_0 = \Phi(z=0)$$

$$\rho = \rho_0 \exp\left(-\frac{\Phi-\Phi_0}{c_s^2}\right)$$

- Poisson equation: $\Delta\Phi = 4\pi G\rho$

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho_0 \exp\left(-\frac{\Phi-\Phi_0}{c_s^2}\right)$$

subst: $\chi = -\dfrac{\Phi-\Phi_0}{c_s^2}\ ,\quad Z_1 = \left(\dfrac{2\pi G\rho_0}{c_s^2}\right)^{1/2} z$

$$\frac{\partial^2 \chi}{\partial Z_1^2} = -2\exp(\chi)\ ;\quad \text{bndy: } \chi = 0 \text{ at } Z = 0$$

$$\frac{\partial \chi}{\partial Z_1} = 0 \text{ at } Z_1 = 0 \quad \left(\text{from plane symmetry}\right)$$

- multiply by $\dfrac{\partial \chi}{\partial Z_1}$:

$$\frac{\partial \chi}{\partial Z_1}\,\frac{\partial^2 \chi}{\partial Z_1^2} = -2\,\frac{\partial \chi}{\partial Z_1}\exp(\chi)$$

$$\frac{1}{2}\,\frac{\partial}{\partial Z_1}\left[\left(\frac{\partial \chi}{\partial Z_1}\right)^2\right] = -2\,\frac{\partial}{\partial Z_1}\left[\exp(\chi)\right]$$

$$\left(\frac{\partial \chi}{\partial Z_1}\right)^2 = C_1 - 4\exp(\chi)\ ;\ \text{from } Z = 0: C_1 = 4 \qquad \boxed{\frac{\partial \chi}{\partial Z_1} = 2\left(1-\exp(\chi)\right)^{1/2}}$$

$$\frac{\partial \chi}{\partial Z_1} = 2\left[1-\exp(\chi)\right]^{1/2}$$

$$2Z_1 + C_3 = \int \frac{d\chi}{\left[1-\exp(\chi)\right]^{1/2}} = \left| \begin{array}{l} s^2 = \exp(\chi) \\ 2s\,ds = \exp(\chi)\,d\chi \end{array} \right| = \int \frac{2\,ds}{s\sqrt{1-s^2}} =$$

$$= \left| \begin{array}{l} s = \sin\varphi \\ ds = \cos\varphi\, d\varphi \end{array} \right| = \int \frac{2\,d\varphi}{\sin\varphi} = \left| \begin{array}{l} t = \operatorname{tg}\frac{\varphi}{2} \\ dt = \frac{d}{d\varphi}\left(\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}\right)\frac{d\varphi}{2} = \frac{\cos^2\varphi/2 + \sin^2\varphi/2}{\cos^2\frac{\varphi}{2}}\,\frac{d\varphi}{2} = \\ \quad = \frac{1}{2}\left(1+t^2\right)d\varphi \\ \sin\varphi = 2\sin\varphi/2\,\cos\varphi/2 = 2\,\frac{\sin\varphi/2}{\cos\varphi/2}\cos^2\varphi/2\,\frac{1+\frac{\sin^2\varphi/2}{\cos^2\varphi/2}}{1+\frac{\sin^2\varphi/2}{\cos^2\varphi/2}} \\ \quad = \frac{2\operatorname{tg}\varphi/2}{1+\operatorname{tg}^2\varphi/2} = \frac{2t}{1+t^2} \end{array} \right|$$

$$= 2\int\frac{dt}{t} = 2\ln t$$

$$\Rightarrow 2Z_1 + C_3 = 2\ln t$$

$$t = C_4 \exp(Z_1)$$

Initial cond:

$\chi(Z_1=0) = 0 \Rightarrow s\big|_{Z_1=0} = 1$

$\Rightarrow \varphi(Z_1=0) = \pi/2 \quad \Rightarrow t(Z_1=0) = 1$

$\Rightarrow C_4 = 1$

- desubstitution

$$\exp(\chi) = s^2 \Rightarrow s = \exp(\chi/2)$$

$$s = \sin\varphi = \frac{2t}{1+t^2} \Rightarrow \exp(\chi/2) = \frac{2t}{1+t^2}$$

$$\Rightarrow t = \exp(Z_1)$$

$$\exp(\chi/2) = \frac{2\exp(Z_1)}{1+\exp(2Z_1)} = \frac{1}{\cosh Z_1}$$

$$\chi = -\frac{\Phi-\Phi_0}{c_s^2}, \quad Z_1 = \left[\frac{2\pi G\rho_0}{c_s^2}\right]^{1/2} z$$

$$\exp\left(-\frac{\Phi-\Phi_0}{c_s^2}\right) = \frac{1}{\cosh^2\left[\left(\frac{2\pi G\rho_0}{c_s^2}\right)^{1/2} z\right]}$$

$$\Phi = 2c_s^2 \ln\left\{\cosh\left[\left(\frac{2\pi G\rho_0}{c_s^2}\right)^{1/2} z\right]\right\} + \Phi_0$$

and using: $\rho = \rho_0 \exp\left[-\frac{\Phi-\Phi_0}{c_s^2}\right]$

$$\underline{\underline{\rho = \rho_0\, \operatorname{sech}^2\left[\left(\frac{2\pi G\rho_0}{c_s^2}\right)^{1/2} z\right]}}$$