Hydrodynamic simulations of the ISM 3

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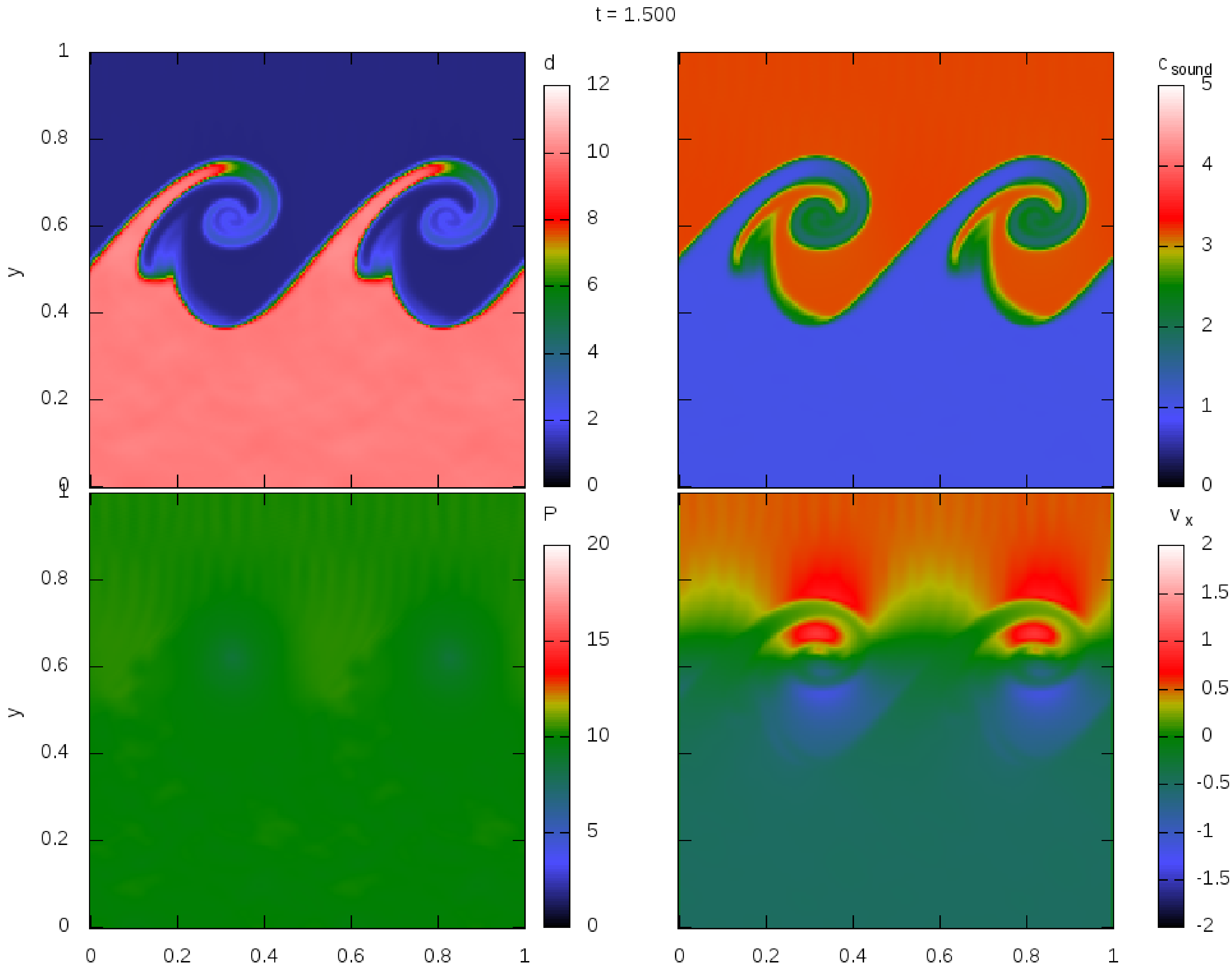
Outline:

1. Homework: Kelvin-Helmholtz instability with ZEUS
2. Self-gravity:
   → FFT
   → Iterative/Multigrid
   → Tree
3. Cooling and heating
   → Isobaric and isochoric thermal instability
4. Radiation transport:
   → ionizing
   → diffuse
5. Magnetic field


t = 1.500
d
c sound
P
v x
y

Self-gravity: Poisson equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho$$

discretization:

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{\frac{\Phi_{i+1,j,k}-\Phi_{i,j,k}}{\Delta x} - \frac{\Phi_{i,j,k}-\Phi_{i-1,j,k}}{\Delta x}}{\Delta x} = \frac{\Phi_{i+1,j,k} - 2\Phi_{i,j,k} + \Phi_{i-1,j,k}}{\Delta x^2}$$

discretize Poisson equation (on a regular grid $\Delta x = \Delta y = \Delta z$):

$$\Phi_{i+1,j,k}+\Phi_{i-1,j,k}+\Phi_{i,j+1,k}+\Phi_{i,j-1,k}+\Phi_{i,j,k+1}+\Phi_{i,j,k+1}-6\Phi_{i,j} = 4\pi G\Delta x^2\rho_{i,j,k}$$

- huge set of linear eqs. (1 equation for each grid cell)
- 1D: tri-diagonal matrix $\Rightarrow$ self-gravity is trivial in 1D
- 2D, 3D: tr-diag. with additional diagonals/fringes

Self-gravity: Spectral methods (FFT)

- only on uniform grids; best with periodic boundary conditions (isolated boundaries hard and inaccurate)

$$\Phi_{j,k} = \frac{1}{N_x N_y} \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} \hat{\Phi}_{m,n} \exp\left(-\frac{2\pi ijm}{N_x}\right) \exp\left(-\frac{2\pi ikn}{N_y}\right)$$

$$\rho_{j,k} = \frac{1}{N_x N_y} \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} \hat{\rho}_{m,n} \exp\left(-\frac{2\pi ijm}{N_x}\right) \exp\left(-\frac{2\pi ikn}{N_y}\right)$$

insert into the discretized Poisson equation:

$$\hat{\Phi}_{m,n} \left[\exp\left(\frac{2\pi im}{N_x}\right) + \exp\left(-\frac{2\pi im}{N_x}\right) + \exp\left(\frac{2\pi in}{N_y}\right) + \exp\left(-\frac{2\pi in}{N_y}\right) - 4\right]$$
$$= 4\pi G \Delta x^2 \hat{\rho}_{m,n}$$

which has a solution:

$$\hat{\Phi}_{m,n} = \frac{\hat{\rho}_{m,n} \Delta x^2}{2\left[\cos\left(\frac{2\pi m}{N_x}\right) + \cos\left(\frac{2\pi n}{N_y}\right) - 2\right]} \qquad \xrightarrow{\text{FFT}} \qquad \Phi_{j,k}$$

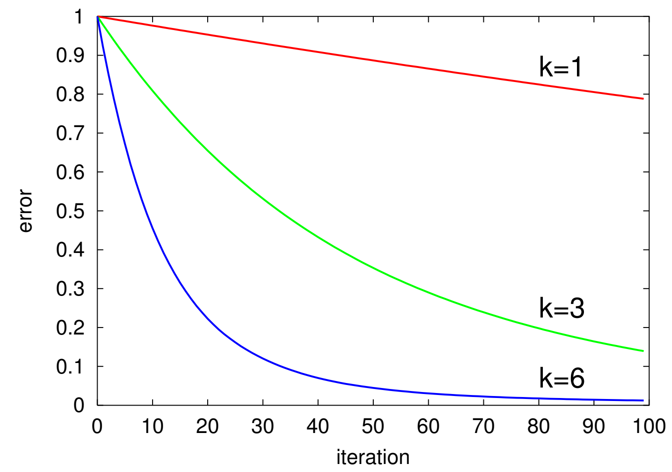
Self-gravity: Iterative methods

- 1D Poisson equation (for simplicity): $\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G\rho$
- discretization: $\Phi_{i+1} + \Phi_{i-1} - 2\Phi_i = 4\pi G\Delta x^2 \rho_i$

- Jacobi method:
  $$\Phi_i^{(1)} = \frac{1}{2}(\Phi_{i-1}^{(0)} + \Phi_{i+1}^{(0)} - 4\pi G\Delta x^2 \rho_i)$$
  $\rightarrow$ **new values are written to the separate array**

- Gauss-Seidel method:
  $$\Phi_i^{(1)} \leftarrow \frac{1}{2}(\Phi_{i-1}^{(0)} + \Phi_{i+1}^{(0)} - 4\pi G\Delta x^2 \rho_i)$$
  $\rightarrow$ **new values overwrite the old ones immediately**

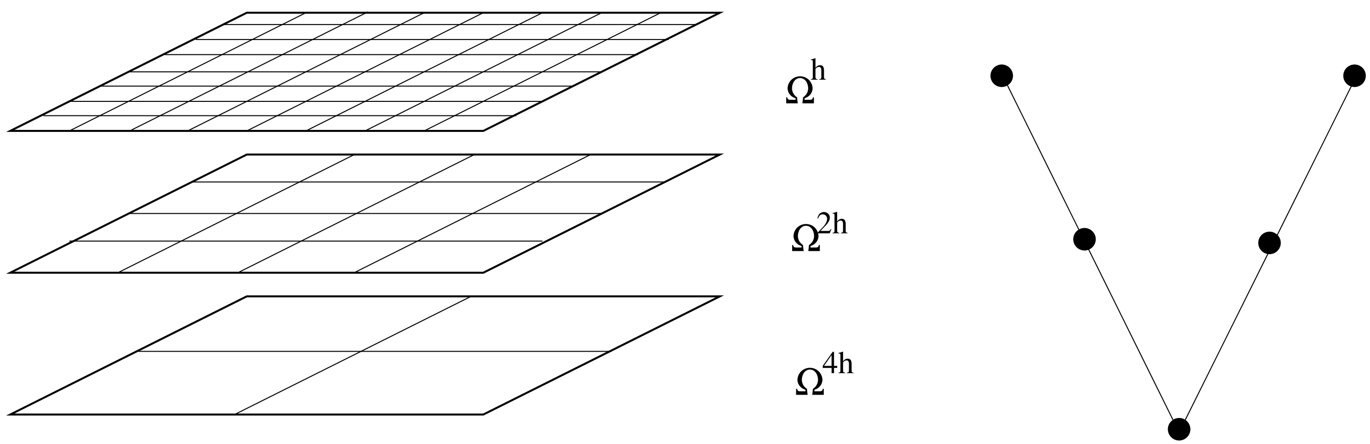
Convergence rate of various Fourier modes of the error:



Problem: standard relaxation methods are very ineffective in reducing the smooth modes of the error.

Self-gravity: Multigrid methods

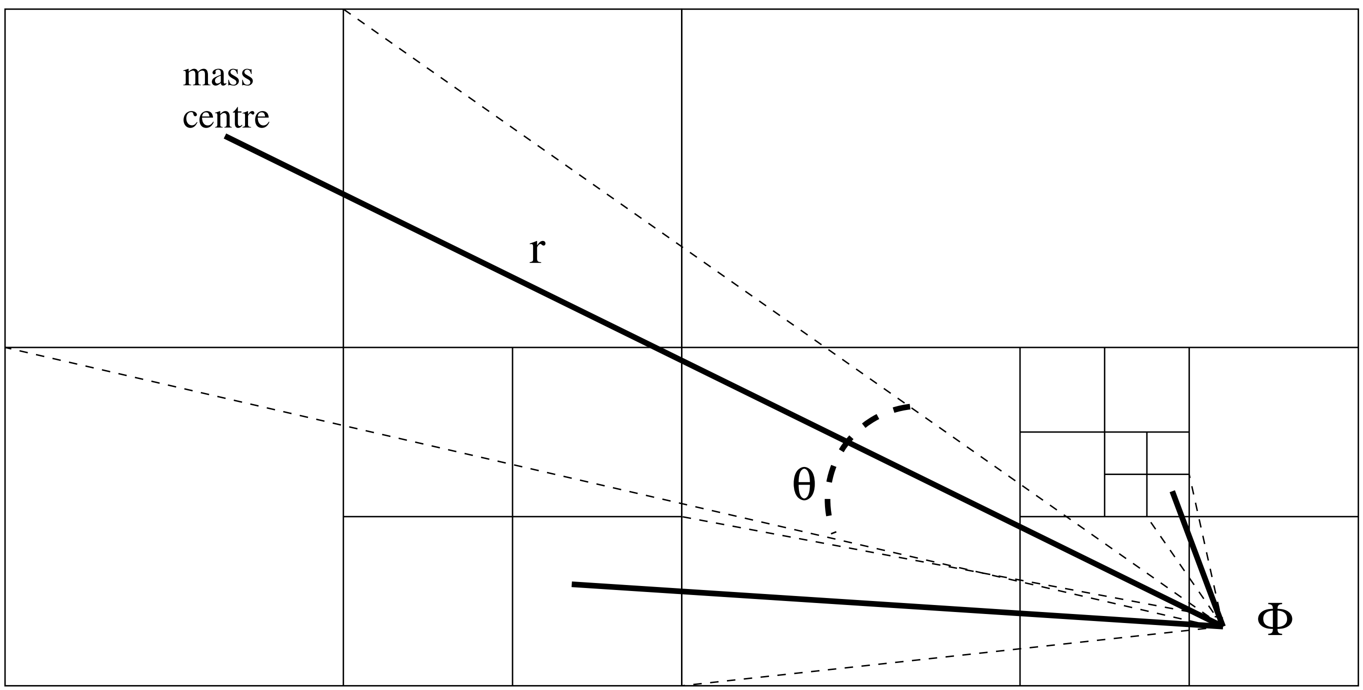
- basic idea:
  long wavelength modes look more oscillatory on a coarser grid
- the V-cycle scheme:
  1. relax on the fine grid
  2. restrict the problem to the coarser grid and do a relaxation
  3. . . .
  4. solve the set of equations on the coarsest grid (exactly)
  5. interpolate to a finer grid and do a relaxation
  6. . . .
  7. final relaxation of the finest grid



Self-gravity: Tree

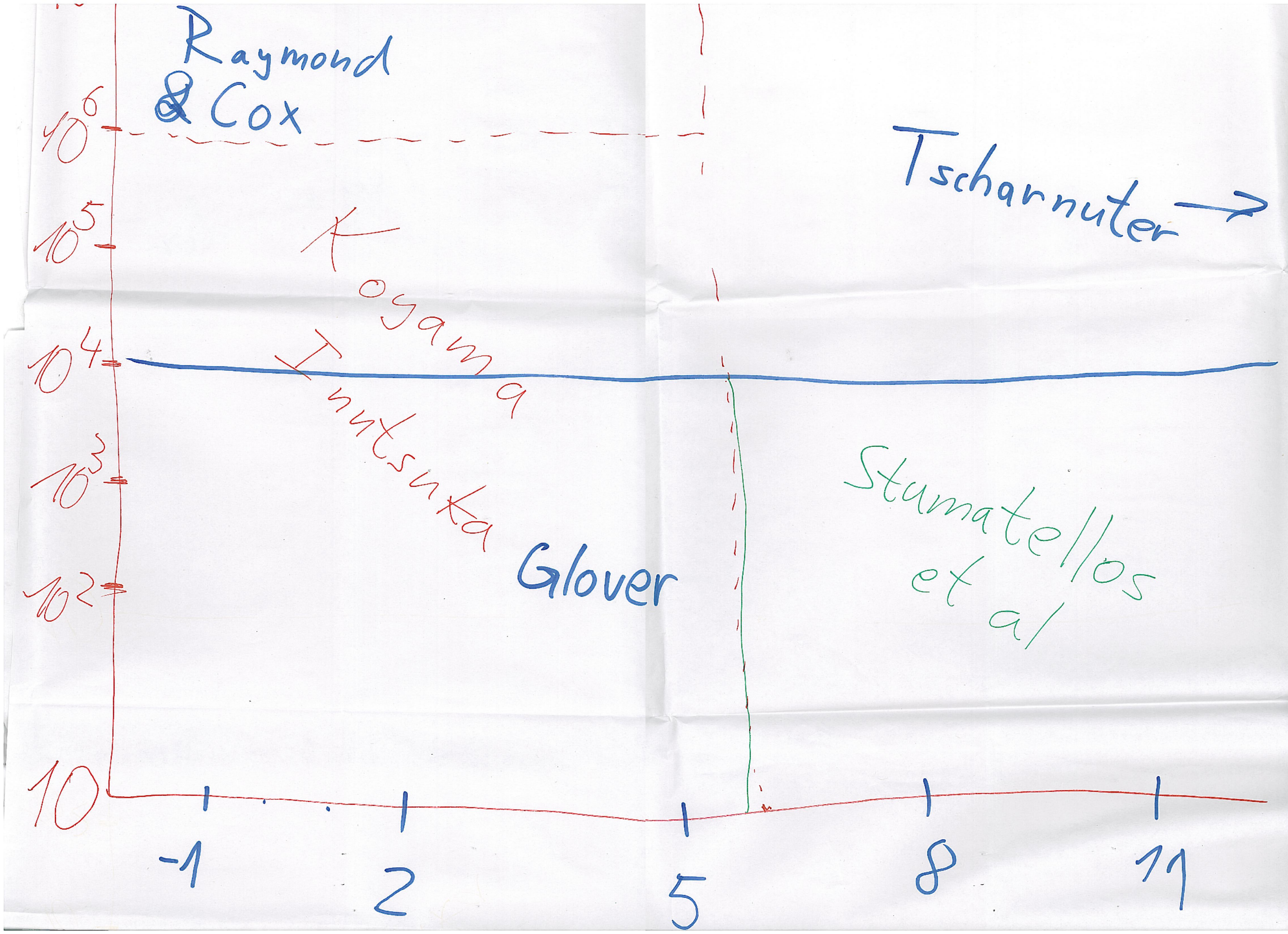
1. Build the tree
   → extends from the whole CD down to individual grid cells (or particles)
   → each node: mass, mass centre (+ optionally higher multipole moments)
2. Tree walk (for each grid cell start with the largest node)
   → if $\theta \equiv \frac{\text{size of the node}}{\text{distance to mass centre}} < \theta_{\text{lim}}$ continue the tree walk opening its children
   → othewise add the node contribution to the potential

$$\Phi \leftarrow \Phi - \frac{GM_{\text{node}}}{r} - \text{higher moment corrections}$$



Cooling and Heating

Zeus et al. Star Forming symposio (Katarini, 5-9th July 2010)

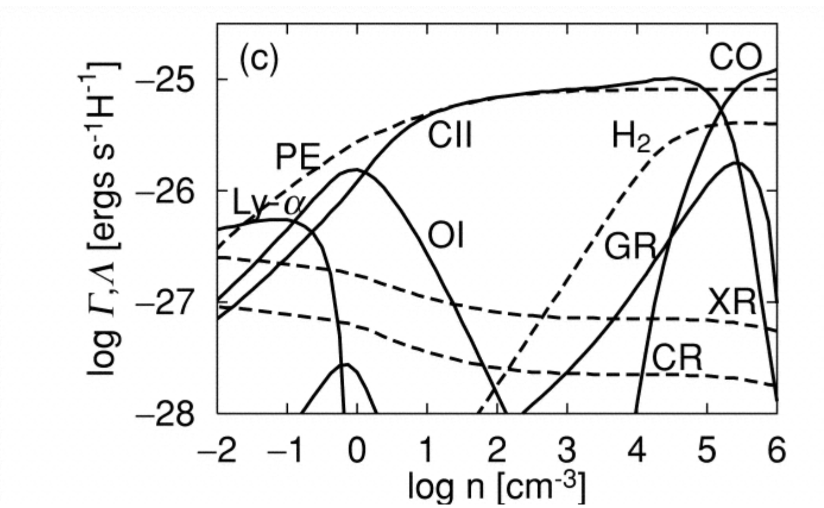


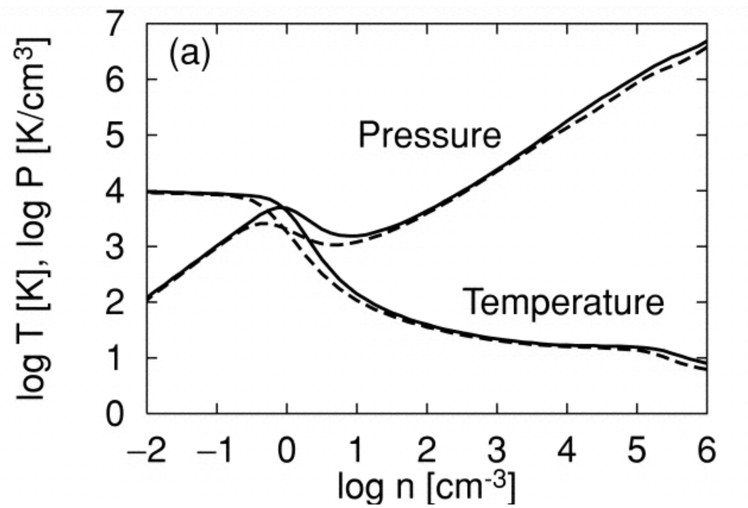
C&H: low densities and low temperatures

- Koyama & Inutsuka, 2000, ApJ, 523, 980; 2002, ApJ, 564, L97
- $n < 10^5$ cm^{-3}, $T < 10^5$ K
- isobaric thermal instability
- analytical fit:

$$\frac{\Lambda(T)}{\Gamma} = 10^7 \exp\left(\frac{-1.184\times 10^5}{T+1000}\right) + 1.4\times 10^{-2}\sqrt{(T)}\exp\left(\frac{-92}{T}\right)\mathrm{cm}^3$$

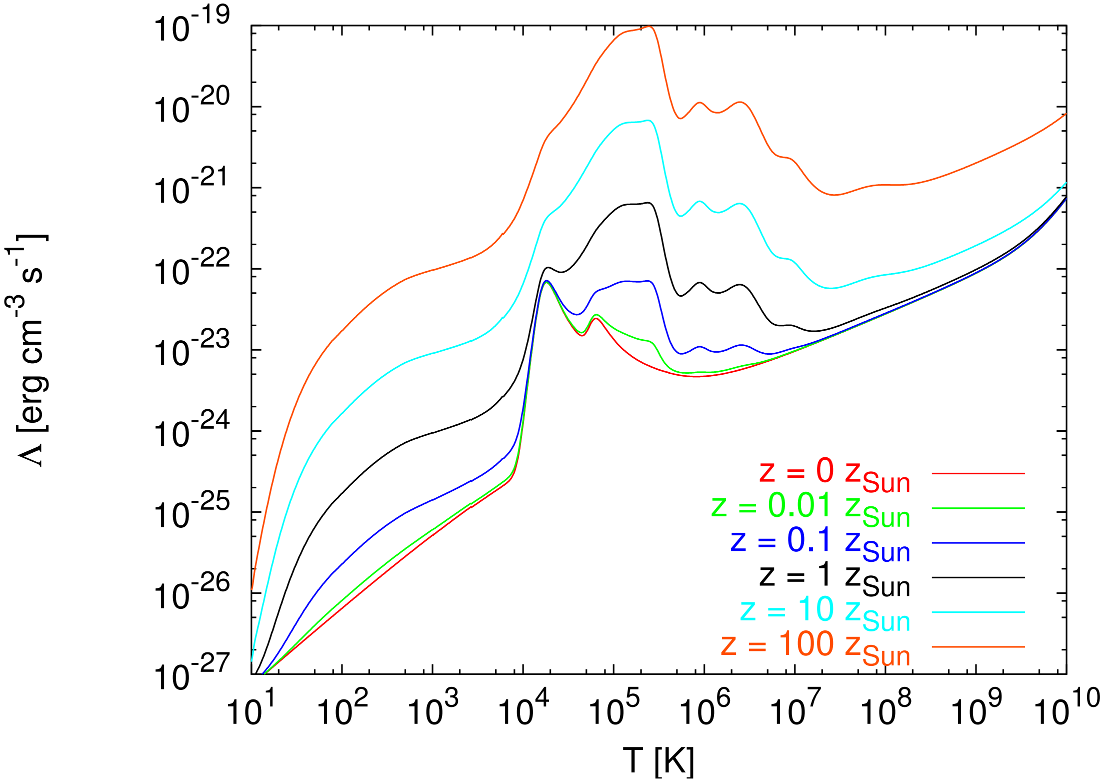
$$\Gamma = 2\times 10^{-26}\mathrm{erg\,s^{-1}}$$





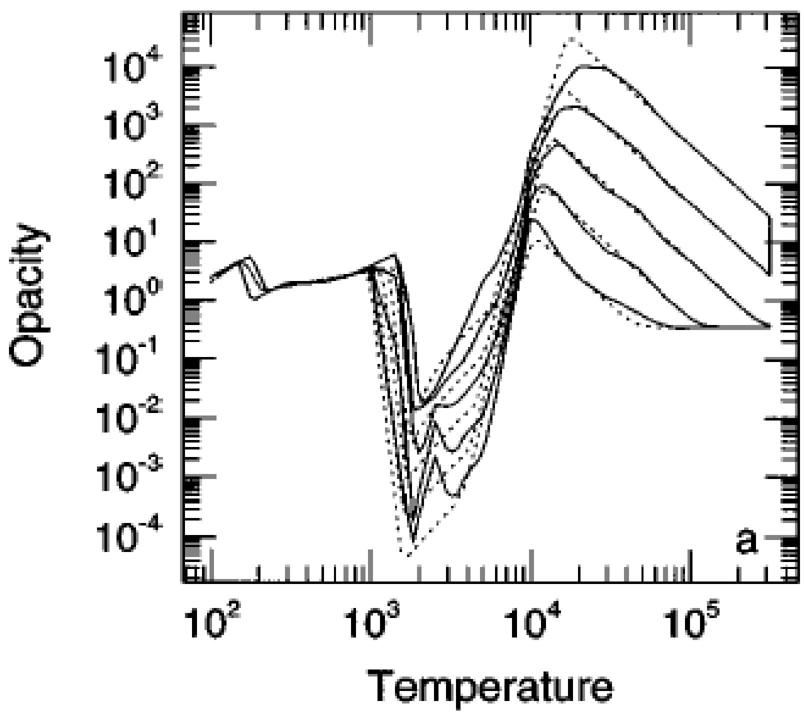
C&H: low densities and high temperatures

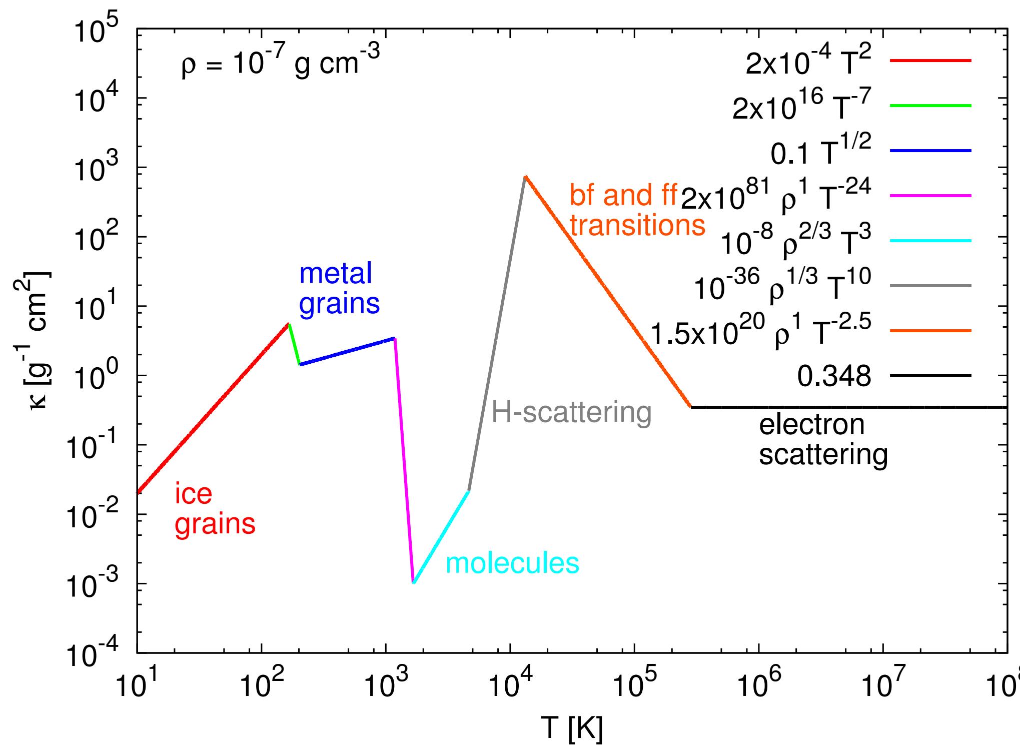
- Raymond and Cox c.f; Sutherland & Dopita, 1993, ApJS, 88, 253
- $n < 10^5$ cm^{-3}, $T > 10^4$ K
- isochoric thermal instability



C&H: high densities

- Bell & Lin, 1994, 427, 987
- $n > 10^5$ cm^{-3}, dust and gas thermally coupled
- optically thin regime: $\frac{de}{dt}\Big|_{\rm RAD} = -4\sigma_{\rm SB}T^4\kappa(\rho,T)$
- optically thick: see Stamatellos et al, 2007, A&A, 475, 37



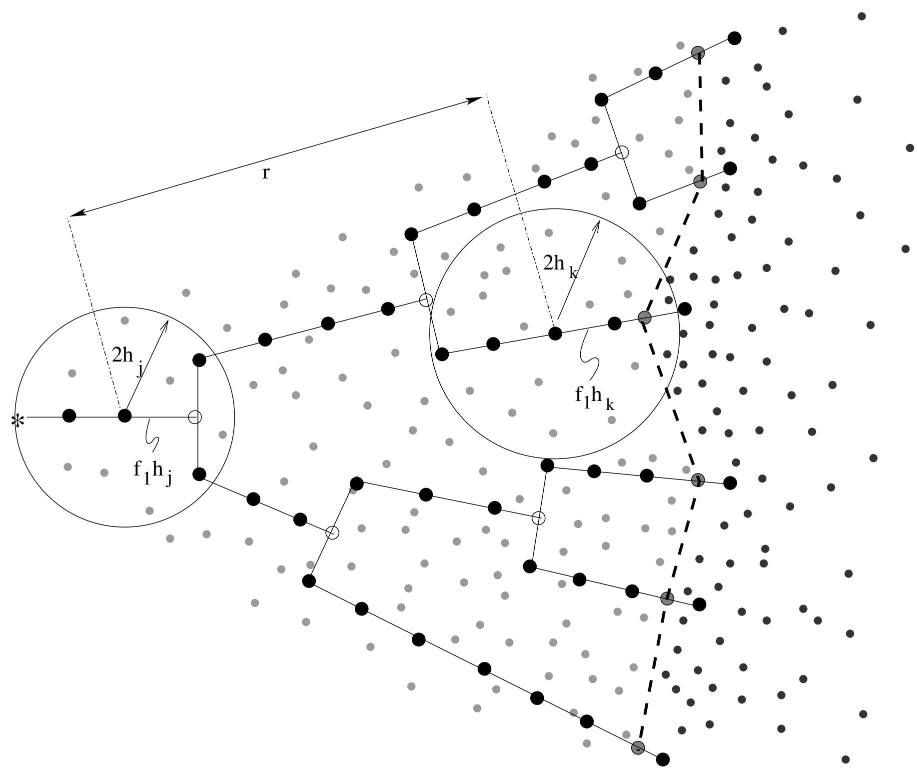


Ionizing radiation

- On-the-spot approximation

$$N_{\rm UV} = 4\pi \int_0^{R_{\rm IF}} \frac{\alpha^\star \rho^2}{(\mu m_{\rm H})^2} r^2 dr, \qquad \alpha^\star \simeq 2.7 \times 10^{-13}\ {\rm cm}^3\,{\rm s}^{-1}$$

- Ray-tracing, see e.g. Bisbas et al., 2009, A&A, 497, 649



Diffuse radiation

- Solving RT generally extremaly hard (7 dimensions: $x, y, z, \theta, \phi, \nu, t$)
- Monte-Carlo codes: see Ercolano et al., 2003, MNRAS, 340, 1136
- Flux Limited Diffusion approximation, (Levermore & Pomraning, 1981, ApJ, 248, 341)

$$\left.\frac{de}{dt}\right|_{\mathrm{rad}} = -\nabla \cdot \mathbf{F}$$

$$\mathbf{F} = -\frac{c\lambda}{\kappa_R \rho}\nabla(aT^4)$$

$$\text{Flux limiter : } \lambda = \frac{2+R}{6+3R+R^2}, \qquad R = \frac{\nabla e}{\kappa_R \rho e}$$

$R \to 0$ (optically thick) $\lambda \to 1/3$: $\quad \mathbf{F} = -\frac{c}{3\kappa_R \rho}\nabla(aT^4)$

$R \to \infty$ (optically thin) $\lambda \to 1/R$: $\quad |\mathbf{F}| = ce$

- FLD is a boundary value problem, needs a Poisson solver

Ideal MHD equations

- assuming large Magnetic Reynolds number: $R_{\rm eM} = LV/\eta_{\rm e}$, i.e. small electric resistinity $\eta_{\rm e}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla P - \frac{1}{8\pi}\nabla(\mathbf{B}^2) + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{v}) = -P\nabla \cdot \mathbf{v}$$

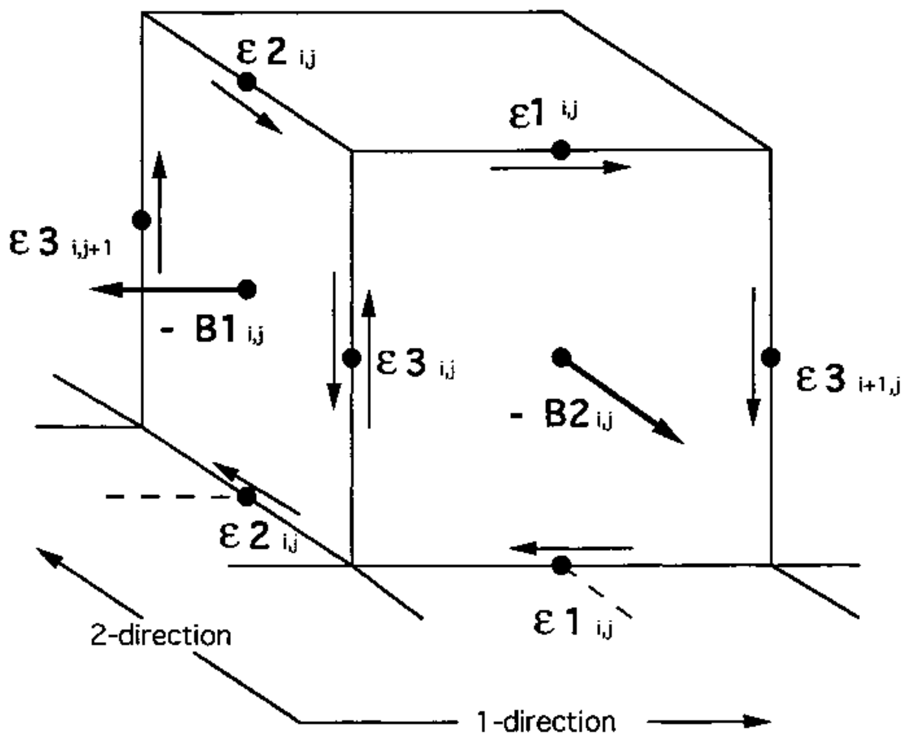
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- Lorentz forces (similar to HD source terms; MoC)
- induction equation (problem to maintain $\nabla \cdot \mathbf{B} = 0$)

Constrained transport

- if $\nabla \cdot \mathbf{B} = 0$ intially, it stays zero down to machine precision
- induction eq. solved in a form:

$$\frac{\partial \Phi_m}{\partial t} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{I} = \oint_C \mathcal{E} \cdot d\mathbf{I}$$



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