FTCS (Forward Time Centered Space)

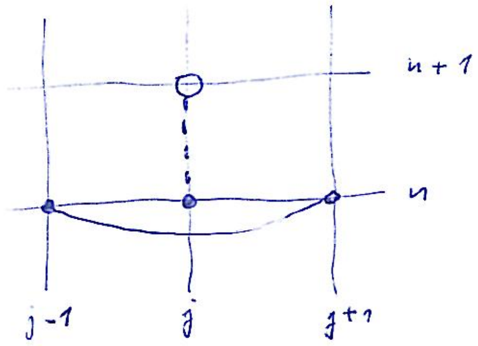
$$\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial x} = 0$$

Grid: $x_j = x_0 + j\Delta x$, $t_n = t_0 + n\Delta t$; q_j^n

Derivatives:

$$\left.\frac{\partial q}{\partial t}\right|_{j,n} = \frac{q_j^{n+1} - q_j^n}{\Delta t} + \mathcal{O}(\Delta t)$$

$$\left.\frac{\partial q}{\partial x}\right|_{j,n} = \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2)$$



FTCS formula:

$$\frac{q_j^{n+1} - q_j^n}{\Delta t} = -v \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x}$$

- Unfortunately, it does not work!

von Neumann Stability Analysis

for slowly varying coefficients, the solution consists of eigen modes of a form:

$$q_j^n = \xi^n e^{ikj\Delta x}$$

, where k - wavenumber, $\xi = \xi(k)$.
i - imaginary unit

inserting it into eq.:

$$\frac{\xi^{n+1} e^{ikj\Delta x} - \xi^n e^{ikj\Delta x}}{\Delta t} = -v \frac{\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}}{2\Delta x} \quad \Big| : \xi^n \; : e^{ikj\Delta x}$$

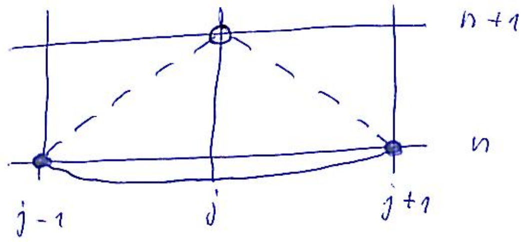
$$\xi(k) - 1 = -\frac{v\Delta t}{2\Delta x} \; e^{ik\Delta x} - e^{-ik\Delta x}$$

$$\xi(k) = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x) \implies |\xi| > 1 \text{ for all } k$$

$\Rightarrow$ FTCS is unconditionally unstable

(stability of num schemes : $<$ unconditionally stable / conditionally stable / uncoditionally unstable)

Lax method



in time derivative:

$$q_j^n \rightarrow \frac{1}{2}\left(q_{j+1}^n + q_{j-1}^n\right)$$

$$\Rightarrow \quad q_j^{n+1} = \frac{1}{2}\left(q_{j+1}^n + q_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(q_{j+1}^n - q_{j-1}^n\right)$$

Stability analysis:

$$\xi^{n+1} e^{ikj\Delta x} = \frac{1}{2}\left[\xi^n e^{ik(j+1)\Delta x} + \xi^n e^{ik(j-1)\Delta x}\right] - \frac{v\Delta t}{2\Delta x}\left[\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right]$$

$$\xi = \frac{1}{2}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - \frac{v\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

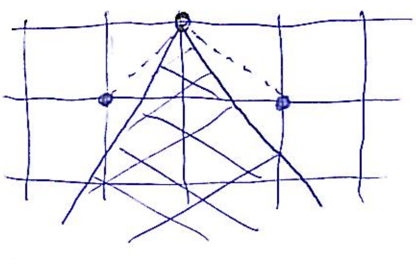
$$\xi(k) = \cos(k\Delta x) - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

$$|\xi| \le 1 \quad \text{if} \quad \frac{|v|\Delta t}{\Delta x} \le 1$$

$$\left(\begin{aligned} &\cos^2(k\Delta x) + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) = \\ &= 1 + \left[\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x) \le 1 \\ &\left[\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x) \le 0 \\ &\Rightarrow \left(\frac{v\Delta t}{\Delta x}\right)^2 \le 1 \end{aligned}\right)$$

- Courant - Friedrichs - Lewy

stable



unstable